Further Hypothesis Tests Cheat Sheet

Edexcel FS2

Variance of a normal distribution

We know that an unbiased estimator of the population variance σ^2 can be calculated using the formula $S^2 = \frac{\sum(X_i-\bar{X})^2}{n-1}$. Clearly different samples give different values in the estimation for σ^s, so s^s is a particular value of the random variable S^2. In order to find a confidence interval for σ^2, we must know the distribution of S^2.

- If a random sample of n observations $X_1, X_2, \ldots, X_n$ is selected from $N(\mu, \sigma^2)$ then $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$ (Percentage points for the chi-squared distribution are given in the formula book, or on some graphical calculators)
- Using the table you can find values of $\frac{(n-1)S^2}{\sigma^2}$ such that the interval between them contains σ^2 with a certain probability:
  - For a probability of α that the variance falls outside the limits, the $100(1-\alpha)\%$ confidence limits are $\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)}$ and $\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}$

Example 1:

A fisherman catches 10 fish from a pink salmon pond. The lengths, x, in cm, of the fish are as follows: 67, 71, 62, 69, 75, 77, 63, 75, 61, 72. Assuming that the length of the fish are normally distributed, find a 90% confidence interval for the variance.

Calculate $\bar{x}$ and s^2, using the formulae or a calculator	$\bar{x} = \frac{67+71+\cdots+72}{10} = 69.2$ $s^2 = \frac{\sum(x_i - 69.2)^2}{9} = 33.5111$
Read off the relevant parts of the chi-squared table- there are 10 measurements so use χ^2_9.	$\chi^2_9(0.95) = 3.325$ $\chi^2_9(0.05) = 16.919$
Using the formulae given above, find the critical points for the variance	$\frac{(n-1)s^2}{\chi^2_{n-1}(0.05)} = \frac{9 \times 33.5111}{16.919} = 17.826$ $\frac{(n-1)s^2}{\chi^2_{n-1}(0.95)} = \frac{9 \times 33.5111}{3.325} = 90.706$
State the confidence interval	The 90% confidence interval for the variance is $(17.826, 90.706)$

Hypothesis testing for the variance of a normal distribution

Hypothesis testing is useful for seeing if the variance has changed. As before, there will be a null and alternative hypothesis, given in terms of the population variance σ^2. To conduct a hypothesis test, we must follow some steps:

1. State the null and alternative hypotheses
2. Specify α, the significance level and v, the degrees of freedom.
3. State the critical region
4. Using the population variance σ^2, and the unbiased estimate s^2 and calculate the value of the test statistic $\frac{(n-1)s^2}{\sigma^2}$.
5. Compare the test statistic and the critical region, stating your conclusion and linking to the context of the question.

Example 2

A farmer is worried that some of his chickens are eating more than their fair share of their feed, and therefore the variance of their weights has increased. The variance is thought to be 0.9kg, and a sample of 8 chickens have weights, in kg, of 2.5, 4.3, 2.1, 4.5, 3, 3.2, 2.7, 4.4. Test at the 2.5% significance level whether the variance has increased.

State your hypotheses	$H_0: \sigma^2 = 0.9$ $H_1: \sigma^2 > 0.9$
Specify the significance level and the degrees of freedom	$\alpha = 5\%$ $v = 8 - 1 = 7$
Find the critical values from the formula book, and the critical region	$\chi^2_7(0.05) = 14.067$ The critical region is $\frac{(n-1)s^2}{\sigma^2} \geq 14.067$
Calculate s^2	$s^2 = 0.8827$
Calculate the test statistic	The test statistic is $\frac{7 \times 0.8827}{0.9} = 6.865$
State your conclusion	$6.865 < 14.067$, so the test statistic is not in the critical region. There is insufficient evidence to reject H_0, so the farmer cannot conclude that some of his chickens are eating more than their fair share of the daily food.

The F-distribution

Retailers like to sell uniform products, so each customer is getting the same thing. As a result of this, the retailer will take note of the mean and the variance of the product and will most likely choose the supplier with the smallest variance. To do this, the F-test is used to determine if two independent random samples are from normal distributions with equal variance.

- For an independent random sample of n_x observations from an $N(\mu_x, \sigma_x^2)$ distribution, and another independent random sample of n_y observations from an $N(\mu_y, \sigma_y^2)$ distribution,
$$\frac{S_x^2/\sigma_x^2}{S_y^2/\sigma_y^2} \sim F_{n_x-1, n_y-1}$$
- If a random sample of n_x observations is taken from a normal distribution with unknown variance σ^2 and independent random sample of n_y observations is taken from a normal distribution with equal but unknown variance, then
$$\frac{S_x^2}{S_y^2} \sim F_{n_x-1, n_y-1}$$

The F-distribution has two parameters v_1 and v_2, so requires big tables- we are interested in the F-distribution for hypothesis testing, so a number of different tables are given, representing each significance level. The tables given critical values which are exceeded with a given probability.

- A critical value $F_{v_1,v_2}(p)$ which is exceeded with probability p is called an upper critical value for p
- The value $F_{v_1,v_2}(1-p)$ for which the observation is less than $F_{v_1,v_2}(1-p)$ with probability p is called a lower critical value for p

The lower critical value for p in F_{v_1,v_2} distribution is the same as the reciprocal of the upper critical value for p in an F_{v_2,v_1} distribution:

$$F_{v_1,v_2}(p) = \frac{1}{F_{v_2,v_1}(1-p)}$$

Example 3:

Find upper critical values for a) $F_{5,8}(0.95)$, b) $F_{8,5}(0.95)$

a) Read off the table the value for $F_{8,5}(0.05)$	$F_{8,5}(0.05) = 4.82$
Use the reciprocal rule to calculate $F_{5,8}(0.95)$	$F_{5,8}(0.95) = \frac{1}{F_{8,5}(0.05)} = \frac{1}{4.82} = 0.21$
b) Read off the table value for $F_{5,8}(0.05)$	$F_{5,8}(0.05) = 3.69$
Use the reciprocal rule to calculate $F_{8,5}(0.95)$	$F_{8,5}(0.95) = \frac{1}{F_{5,8}(0.05)} = \frac{1}{3.69} = 0.27$

This example highlights how important the order of v_1, v_2 is.

Example 4:

The random variable X follows an F-distribution with 12 and 10 degrees of freedom. Find $P\left(\frac{1}{4.30} < X < 2.91\right)$

Looking at the upper tail:	$P(X > 2.91) = P(F_{12,10} > 2.91)$
Notice that from the table, $F_{12,10}(5\%) = 2.91$	$P(X > 2.91) = 0.05$ $P(X < 2.91) = 1 - 0.05 = 0.95$
Looking at the lower tail:	$P\left(X < \frac{1}{4.3}\right) = P\left(F_{12,10} < \frac{1}{4.3}\right)$
Use the rule $P(F_{v_1,v_2} < f) = P\left(\frac{1}{f} < F_{v_2,v_1}\right)$	$P\left(F_{12,10} < \frac{1}{4.3}\right) = P(F_{10,12} > 4.3)$
From the table, $F_{10,12}(1\%) = 4.3$, so:	$P\left(X < \frac{1}{4.3}\right) = 0.01$
Calculate the interval	$P\left(\frac{1}{4.30} < X < 2.91\right) = P(X < 2.91) - P\left(X < \frac{1}{4.3}\right)$ $= 0.95 - 0.01 = 0.94$

The F-test

If two samples are taken from two populations with normal distributions, the F-test is used to see if the two variances are equal, and can be completed by the following steps:

1. Find s_x^2, the larger variance, and s_y^2, the smaller variance.
2. State the null hypothesis- $H_0: \sigma_x^2 = \sigma_y^2$
3. State the alternative hypothesis- $H_1: \sigma_x^2 > \sigma_y^2$ for a one-tailed test, or $H_1: \sigma_x^2 \neq \sigma_y^2$ for a two-tailed test
4. Find the critical value of F_{v_x,v_y} in the tables, where v_x is the degrees of freedom of the distribution with the larger variance and v_y is the degrees of freedom of the distribution with the smaller variance.
5. Write down the critical region
6. Calculate the test statistic: $F_{test} = \frac{s_x^2}{s_y^2}$
7. Decide whether F_{test} lies in the critical region and state your conclusion. Link back to the context of the question.

Example 5:

Two samples of sizes 6 and 13 ae taken from normal distributions X and Y with variances σ_x^2 and σ_y^2 respectively. The two samples give values $s_x^2 = 7$ and $s_y^2 = 26$. Test, at the 5% level of significance whether the variance of Y is greater than the variance of X.

State the null hypothesis	$H_0: \sigma_x^2 = \sigma_y^2$
State the alternative hypothesis	$H_1: \sigma_x^2 < \sigma_y^2$
Calculate the degrees of freedom	$v_x = 6 - 1 = 5$ $v_y = 13 - 1 = 12$
Find the critical value from the table (remember the order of the distributions matters)	$F_{12,5}(0.05) = 4.68$
Calculate the test statistic- the larger variance divides the smallest	$F_{test} = \frac{s_y^2}{s_x^2} = \frac{26}{7} = 3.714$
State the conclusion to the test	$3.714 < 4.68$ so there is insufficient evidence to reject H_0

Example 6:

Two samples of sizes 25 and 17 are taken from two normal distributions X and Y with variances σ_x^2 and σ_y^2 respectively. The two samples give values $s_x^2 = 8.3$ and $s_y^2 = 3.1$. Test, at the 10% level of significance, whether the variances are different.

State the null hypothesis	$H_0: \sigma_x^2 = \sigma_y^2$
State the alternative hypothesis	$H_1: \sigma_x^2 \neq \sigma_y^2$
Calculate the degrees of freedom	$v_x = 25 - 1 = 24$ $v_y = 17 - 1 = 16$
Find the critical value from the table- notice that this is a two-tailed test, there is still only one critical value, but the p-value is halved.	$F_{24,16}(0.05) = 2.24$
Calculate the test statistic	$F_{test} = \frac{s_x^2}{s_y} = \frac{8.3}{3.1} = 2.677$
State the conclusion to the test	$2.24 < 2.677$, so there is sufficient evidence to reject H_0

Example 7:

Two lightbulb manufacturers A and B produce their lightbulbs in the same factory, who claim that the lives of the lightbulbs have the same variance. However, company B have done their own tests, and believe that their lightbulbs are more unreliable and have a greater variance than company A. They tested a random sample of 11 bulbs from company A, and their tests gave $\Sigma x = 11820, \Sigma x^2 = 12811000$, where x represents the lifetime of a bulb in hours.

When testing a sample of 11 of their own bulbs, they had lifetimes, in hours, of:

938, 1340, 1220, 960, 1138, 1199, 999, 1115, 979, 1056, 1390

Conduct a test at the 5% significance level, whether the variance of the lifetime of the bulbs from company B is greater than the lifetime of the bulbs from company A.

Find the unbiased estimates for the two variances.	$s_A^2 = \frac{1}{n-1}\Sigma x^2 - \frac{1}{n^2-n}(\Sigma x)^2$ $= \frac{1}{10}(12811000) - \frac{1}{11^2-11}(11820)^2$ $= 10987.27$ $s_B^2 = 23521.437$ (from calculator)
State the null hypothesis	$H_0: \sigma_A^2 = \sigma_B^2$
State the alternative hypothesis	$H_1: \sigma_A^2 < \sigma_B^2$
Calculate the degrees of freedom	$v_A = 11 - 1 = 10$ $v_B = 11 - 1 = 10$
Find the critical value	$F_{10,10}(0.05) = 2.98$
Calculate the test statistic	$F_{test} = \frac{s_B^2}{s_A^2} = \frac{23521.437}{10987.27} = 2.140$
State the calculation of the test and link back to the context of the question	$2.14 < 2.98$, so there is insufficient evidence to reject H_0. There is insufficient evidence to suggest that the variance of company B's lightbulbs have a greater variance and are therefore more unreliable.