Probability Generating Functions Cheat Sheet

Edexcel FS1

A probability generating function is a mathematical function that is very useful for dealing with **discrete distributions which take non-negative integer values** (e.g. binomial or Poisson). A probability generating function can be used to generate all of the probabilities within a distribution. More notably, it can also be used to easily find probabilities that pertain to a sum of random variables (i.e. probabilities of the form $P(X+Y=k)$).

The fundamentals

- If a discrete random variable X has probability mass function $P(X=x)$, then the probability generating function of X is given by

$$\boldsymbol{G_X(t)=\sum P(X=x)t^x}$$

For example, consider the discrete probability distribution X:

x	0	1	2	3
$P(X=x)$	0.1	0.2	0.3	0.4

You can see that the coefficients of t^x are the probabilities $P(X=x)$

Then the probability generating function of X is $G_X(t)=0.1t^0+0.2t^1+0.3t^2+0.4t^3$.

- The coefficients of t^x are the probabilities $P(X=x)$.
- For any probability generating function, $G_X(1)=1$.
- The probability generating function for X is also given by $G_X(t)=E(t^X)$.

Example 1: The probability generating function of a discrete random variable X is given by $G_x(t)=k(1+2t+2t^2)^2$.
(a) Find the value of k.
(b) Find $P(Y=1)$.

a) Use the fact that $G_x(1)=1$.	$G_x(1)=k(1+2+2)^2=25k$ $\Rightarrow 25k=1$
Divide through by 25.	$k=\frac{1}{25}$
b) $P(Y=1)$ is given by the coefficient of t in the expansion of $G_x(t)$.	$G_x(t)=\frac{1}{25}(1+2t+2t^2)(1+2t+2t^2)$ $=\frac{1}{25}(1+4t+8t^2+8t^3+4t^4)$ $=\frac{1}{25}+\frac{4}{25}t+\frac{8}{25}t^2+\frac{8}{25}t^3+\frac{4}{25}t^4$
$P(X=1)=$ coefficient of t.	$\therefore P(X=1)=\frac{4}{25}$

Probability generating functions of standard distributions

You need to be able to use the probability generating functions for the poisson, binomial, negative binomial and geometric distributions.

- If a discrete random variable $X\sim B(n,p)$, the p.g.f of X is given by $\mathbf{G_x(t)=(1-p+pt)^n}$
- If a discrete random variable $X\sim NB(r,p)$, then the p.g.f o f X is given by $\mathbf{G_x(t)=\left(\frac{pt}{1-(1-p)t}\right)^r}$
- If a discrete random variable $X\sim Po(\lambda)$, then the p.g.f of X is given by $\mathbf{G_x(t)=e^{\lambda(t-1)}}$
- If a discrete random variable $X\sim Geo(p)$, then the p.g.f of X is given by $\mathbf{G_x(t)=\frac{pt}{1-(1-p)t}}$

These results are given to you in the formula booklet, but you also need to be able to prove them from first principles. To do so, you will need to use the definition of a probability generating function, $G_x(t)=\sum P(X=x)t^x$, and substitute $P(X=x)$ with the probability mass function for whichever distribution you are dealing with.

Binomial distribution

Example 2: The random variable $X\sim B(n,p)$.
Prove, from first principles, that the probability generating function of X is given by $G_x(t)=(1-p+pt)^n$.

Using $G_x(t)=\sum P(X=x)t^x$: We have a sum with limits $x=0$ and $x=n$ since X can only take values between 0 and n inclusive.	$P(X=x)=\binom{n}{x}(p)^x(1-p)^{n-x}$ $\therefore G_x(t)$ $=\sum_{x=0}^{n}\binom{n}{x}(p)^x(1-p)^{n-x}t^x$
Since p and t are raised to the same power, we can rewrite this as:	$G_x(t)=\sum_{x=0}^{n}\binom{n}{x}(pt)^x(1-p)^{n-x}$
But notice that this expression we have now is exactly the expansion of $(a+b)^n$, with $a=pt$ and $b=1-p$. This completes the proof.	but $\sum_{x=0}^{n}\binom{n}{x}(pt)^x(1-p)^{n-x}$ $=(pt+1-p)^n$ $\therefore G_x(t)=(1-p+pt)^n$ as required.

Poisson distribution

Example 3: The random variable $X\sim Po(\lambda)$. Prove, from first principles, that the probability generating function of X is given by $G_x(t)=e^{\lambda(t-1)}$.

Using $G_x(t)=\sum P(X=x)t^x$: We have an infinite sum since a Poisson distributed random variable can theoretically take any non-negative integer value.	$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}$ $\therefore G_x(t)=\sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}t^x$
Since λ and t are raised to the same power, we can rewrite this as:	$G_x(t)=\sum_{x=0}^{\infty}\frac{e^{-\lambda}(\lambda t)^x}{x!}$
Since $(\lambda t)^x$ is independent of x, we can take it outside the sum:	$=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda t)^x}{x!}$
But notice that $\sum_{x=0}^{\infty}\frac{(\lambda t)^x}{x!}$ is the Maclaurin expansion of e^x, just with x replaced by λt.	$e^n=\sum_{x=0}^{\infty}\frac{n^x}{x!}$, so $e^{\lambda t}=\sum_{x=0}^{\infty}\frac{(\lambda t)^x}{x!}$ $\therefore G_x(t)=e^{-\lambda}e^{\lambda t}=e^{\lambda(t-1)}$.

Geometric distribution

Example 4: The random variable $X\sim Geo(p)$. Prove, from first principles, that the probability generating function of X is given by $G_x(t)=\frac{pt}{1-(1-p)t}$.

Using $G_x(t)=\sum P(X=x)t^x$: We have an infinite sum since a geometrically distributed random variable can theoretically take any non-negative integer value.	$P(X=x)=p(1-p)^{x-1}$ $\therefore G_x(t)=\sum_{x=1}^{\infty}p(1-p)^{x-1}t^x$
Expanding the first few terms of the sum allows us to deduce that we are dealing with a geometric series.	$G_x(t)=p[t+(1-p)^1t^1$ $+(1-p)^2t^2+\cdots]$
We have a geometric series with $a=p(1-p)t$ and $r=(1-p)t$. Our sum is an infinite sum so using the sum to infinity for a geometric series will give us the p.g.f.	$\therefore G_x(t)=S_\infty=\frac{a}{1-r}$ $=\frac{pt}{1-(1-p)t}$ as required.

Negative binomial distribution

Example 5: The random variable $X\sim Negative\ B(r,p)$. Prove, from first principles, that the probability generating function of X can be written as $G_x(t)=\left(\frac{pt}{1-(1-p)t}\right)^r$.
You may quote the following result without proof:

$$\sum_{x=r}^{\infty}\binom{x-1}{r-1}q^{x-r}=(1-q)^{-r}\text{ where }q=1-p.$$

Use $G_x(t)=\sum P(X=x)t^x$. We have an infinite sum since a NB distributed random variable can theoretically take any non-negative integer value that is greater than or equal to r.	$P(X=x)=\binom{x-1}{r-1}(p)^r(q)^{x-r}$ $\therefore G_x(t)=\sum_{x=r}^{\infty}\binom{x-1}{r-1}(p)^r(q)^{x-r}t^x$
We can rewrite t^x as $t^{x-r}t^r$, so that the powers of t match those of p and q.	$G_x(t)=\sum_{x=r}^{\infty}\binom{x-1}{r-1}(p)^r(q)^{x-r}t^{x-r}t^r$
Now bringing the terms with equal powers together:	$=\sum_{x=r}^{\infty}\binom{x-1}{r-1}(pt)^r(qt)^{x-r}$
Since $(pt)^r$ is independent of x, we can take it out of the sum:	$=(pt)^r\sum_{x=r}^{\infty}\binom{x-1}{r-1}(qt)^{x-r}$
Now we can use the result we were given in the question but replacing q with qt.	$\sum_{x=r}^{\infty}\binom{x-1}{r-1}(qt)^{x-r}=(1-qt)^{-r}$ $\therefore G_x(t)=(pt)^r(1-qt)^{-r}$ $=\left(\frac{pt}{1-(1-p)t}\right)^r$

Mean and variance of a distribution

You can differentiate the probability generating function to find the mean and variance for a probability distribution.

- $E(X)=G'_x(1)$
- $Var(X)=G''_x(1)+G'_x(1)-\left(G'_x(1)\right)^2$

The following fact is also useful:

- $\frac{1}{n!}G^n{}_x(0)=P(X=n)$

This means that substituting $t=0$ into the n^{th} derivative of $G_X(t)$ will give you $P(X=n)$.

Example 6: The probability generating function of a discrete random variable X is given by $G_x(t)=\frac{t^2(2+t)^4}{81}$.
Find the mean and variance of X.

We first differentiate with respect to t. We need to make use of the product and chain rules here.	$G'_x(t)=\frac{2t(2+t)^4}{81}+\frac{4t^2(2+t)^3}{81}$
To find the mean, substitute $t=1$.	$G'_x(1)=\frac{2(2+1)^4}{81}+\frac{4(2+1)^3}{81}=\frac{10}{3}$ $\therefore E(X)=\frac{10}{3}$
Now to find the variance, we first need to find $G''_x(1)$.	$G''_x(t)=\frac{2(2+t)^4}{81}+\frac{8t(2+t)^3}{81}+\frac{8t(2+t)^3}{81}+\frac{24t(2+t)^3}{81}$
Substitute $t=1$.	$\Rightarrow G''_x(1)=\frac{2(2+1)^4}{81}+\frac{8(2+1)^3}{81}+\frac{8t(2+1)^3}{81}+\frac{24t(2+1)^3}{81}=\frac{26}{3}$
Use $Var(X)=G''_x(1)+G'_x(1)-\left(G'_x(1)\right)^2$	$\therefore Var(X)=\frac{26}{3}+\frac{10}{3}-\left(\frac{10}{3}\right)^2=\frac{8}{9}$

Example 7: The discrete random variable X has probability generating function $G_x(t)=\frac{at}{b-t^2}$, where a and b are positive constants.
Given that the mean of X is 1.5, find the values of a and b.

We first differentiate with respect to t. We need to make use of the quotient (or product) rule here.	$G'_x(t)=\frac{a(b-t^2)+2at^2}{(b-t^2)^2}$
The mean is 1.5, so substitute $t=1$ and equate to 1.5.	$G'_x(1)=\frac{a(b-1)+2a}{(b-1)^2}=1.5$
Rearrange to make a the subject. Name this equation [1].	$1.5(b-1)^2=ab-a+2a$ $\therefore a(b+1)=1.5(b-1)^2$ $\Rightarrow a=\frac{1.5(b-1)^2}{(b+1)}$ [1]
We also know that $G_x(1)=1$.	$\Rightarrow G_x(1)=\frac{a}{b-1}=1$ $a=b-1$ [2]
Now we just need to solve [1] and [2] simultaneously. Substitute [2] into [1].	$b-1=\frac{1.5(b-1)^2}{(b+1)}$ $(b-1)(b+1)=1.5(b-1)^2$
Solve for b.	$(b-1)[b+1-1.5(b-1)]=0$ $(b-1)(-0.5b+2.5)=0$
Multiply through by 2.	$(b-1)(5-b)=0$ So $b=5$ or $b=1$. But if $b=1$, then $a=0$ which cannot be a solution since we are told that a is positive. That means $b=5$ and $a=5-1=4$.

Sums of independent random variables

- If X and Y are independent random variables with probability generating functions $\boldsymbol{G_X(t)}$ and $\boldsymbol{G_Y(t)}$, then the probability generating function of $Z=X+Y$ is given by:

$$\boldsymbol{G_Z(t)=G_X(t)\times G_Y(t)}$$

- If $Z=X+Y$, then $E(Z)=E(X)+E(Y)$.

You are given this result and you do not need to be able to prove it.

Example 8: A random variable X has a probability generating function $G_X(t)=\frac{4}{(3-t)^2}$.
A second random variable Y has a probability generating function $G_Y(t)=\frac{t}{(3-2t)^3}$.
Given that X and Y are independent,
(a) write down the probability generating function for $Z=X+Y$.
(b) find $E(Z)$.

a) We multiply the probability generating functions together to obtain the p.g.f of Z.	$G_Z(t)=G_X(t)\times G_Y(t)$ $G_Z(t)=\frac{4}{(3-t)^2}\times\frac{t}{(3-2t)^3}=\frac{4t}{(3-t)^2(3-2t)^3}$
b) To find the mean, we can use $E(Z)=E(X)+E(Y)$. So, we start by finding $E(X)$.	$G'_X(t)=4(3-t)^{-2}$ $G'_X(1)=4(3-1)^{-2}=1=E(X)$
Find $E(Y)$.	$G'_Y(t)=6t(3-2t)^{-4}$ $G'_Y(1)=6(3-2)^{-4}=6=E(Y)$
Use $E(Z)=E(X)+E(Y)$.	$\Rightarrow E(Z)=E(X)+E(Y)=1+6=7$

Linear transformations of random variables

- If the discrete random variable X has probability generating function $\boldsymbol{G_X(t)}$, then the probability generating function of the discrete random variable $\boldsymbol{Y=aX+b}$ is given by

$$\boldsymbol{G_Y(t)=t^bG_X(t^a)}$$

You are not given this result. You do not need to be able to prove it.

Example 9: A random variable X has a probability generating function $G_x(t)=\frac{1}{2}t+\frac{1}{2}t^3$.
Find the probability generating functions for the following random variables:
a) $Y=3X$
b) $Y=2X+3$
c) $Y=4X-5$

a) Use $\boldsymbol{G_Y(t)=t^bG_X(t^a)}$ with $a=3,b=0$.	$G_Y(t)=t^0G_X(t^3)=\frac{1}{2}t^3+\frac{1}{2}(t^3)^3=\frac{1}{2}t^3+\frac{1}{2}t^9$
b) Use $\boldsymbol{G_Y(t)=t^bG_X(t^a)}$ with $a=2,b=3$.	$G_Y(t)=t^3G_X(t^2)=t^3\left(\frac{1}{2}t^2+\frac{1}{2}(t^2)^3\right)=\frac{1}{2}t^5+\frac{1}{2}t^9$
c) Use $\boldsymbol{G_Y(t)=t^bG_X(t^a)}$ with $a=4,b=-5$.	$G_Y(t)=t^{-5}G_X(t^4)=t^{-5}\left(\frac{1}{2}t^4+\frac{1}{2}(t^4)^3\right)=\frac{1}{2}t^{-1}+\frac{1}{2}t^7$

Example 10: The discrete random variable $X\sim B(n,p)$ has probability generating function given by $\boldsymbol{G_X(t)=(0.4+0.6t)^2}$

a) Write down the values of n and p.
Two independent observations X_1 and X_2 are taken from the distribution of X. The random variable $Y=X_1+X_2$.
b) Use calculus to show that $E(Y^2)=6.72$.

a) Use the result that a binomially distributed random variable has p.g.f $\mathbf{(1-p+pt)^n}$.	$n=2,p=0.6$
b) Find the p.g.f of Y. This is found by multiplying the p.g.f's of X_1 and X_2, which are both the same since they are from the distribution of X.	$G_Y(t)=G_{X_1}(t)\times G_{X_2}(t)=[G_X(t)]^2$ $\therefore G_Y(t)=(0.4+0.6t)^4$
Rearranging $Var(Y)=E(Y^2)-E(Y)^2$ gives $E(Y^2)=Var(Y)+E(Y)^2$. So, we need to find $E(Y)$ and $Var(Y)$ in order to find $E(Y^2)$.	$G'_Y(t)=4(0.6)(0.4+0.6t)^3$ $E(Y)=G'_Y(1)=4(0.6)(0.4+0.6)^3=2.4$ $G''_Y(t)=4(3)(0.6)^2(0.4+0.6t)^2$ $G''_Y(1)=4(3)(0.6)^2(0.4+0.6)^2=4.32$
Use $Var(Y)=G''_Y(1)+G'_Y(t)-G'_Y(t)^2$.	$\therefore Var(Y)=4.32+2.4-2.4^2=0.96$
Use $E(Y^2)=Var(Y)+E(Y)^2$.	$E(Y^2)=0.96+2.4^2=6.72$