Discrete Random Variables Cheat Sheet

What is a discrete random variable?

A discrete random variable is a variable whose outcome depends on a random event. We call it discrete because it can only take on specific values in a given range.

The probability distribution for a discrete random variable is usually given in tabular form. For example, if X represents the score when a fair six-sided dice is rolled, then the probability distribution would look like:

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Uppercase is used to denote the random variable while lowercase is used to denote observations of the random variable.

The sum of all probabilities should equal 1.

Finding the expected value of a discrete random variable

The expected value of a random variable, denoted as $E(X)$, is its theoretical average value. If you were to take a number of observations from X, you could find the mean of these observations. The greater the number of observations taken, the closer your sample mean will be to the expected value. You need to be able to calculate the expected value of a discrete random variable.

- $E(X) = \sum xP(X=x)$

In general, $[E(X)]^2 \neq E(X^2)$

If X is a discrete random variable then it follows that X^2 is also a discrete random variable. You may also need to find the expected value of X^2, denoted as $E(X^2)$.

- $E(X^2) = \sum x^2 P(X=x)$

Example 1: A discrete random variable X has the following probability distribution.

x	1	2	3	4
$P(X=x)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

a) Find $E(X)$.
b) Find $E(X^2)$.

We apply the formula for $E(X)$. Sum up the product of each x value and the associated probability.	a) $E(X) = 1\left(\frac{12}{25}\right) + 2\left(\frac{6}{25}\right) + 3\left(\frac{4}{25}\right) + 4\left(\frac{3}{25}\right) = 1.92$
We need to apply the formula for $E(X^2)$. Before we can do this however, we need to create a new row in our table for x^2.	x^2: 1, 4, 9, 16 x: 1, 2, 3, 4 $P(X=x)$: $\frac{12}{25}$, $\frac{6}{25}$, $\frac{4}{25}$, $\frac{3}{25}$
Now we can apply the formula.	b) $E(X^2) = 1\left(\frac{12}{25}\right) + 4\left(\frac{6}{25}\right) + 9\left(\frac{4}{25}\right) + 16\left(\frac{3}{25}\right) = 4.8$

Some questions will also require you to use the fact that the sum of the probabilities for a discrete random variable must equate to 1:

- $\sum_{\text{all } x} P(X=x) = 1$

Example 2: The random variable Y has the following probability distribution:

y	1	2	3	4	5
$P(Y=y)$	0.1	a	b	0.2	0.1

Given that $E(Y) = 2.9$, find the value of a and b.

We start by finding $E(Y)$:	$E(Y) = 1(0.1) + 2(a) + 3(b) + 4(0.2) + 5(0.1)$ $= 1.4 + 2a + 3b$
Equating to 2.9 and simplifying: We label this equation [1].	$\therefore 1.4 + 2a + 3b = 2.9$ $2a + 3b = 1.5$ [1]
Now we use the fact that the total probability is equal to 1:	$0.1 + a + b + 0.2 + 0.1 = 1$ $\Rightarrow a + b = 0.6$ [2]
To find a *and* b, solve [1] and [2] simultaneously:	[2] × 2: $2a + 2b = 1.2$ Subtract this equation from [1]: $2a + 3b - 2a - 2b = 1.5 - 1.2$ $b = 0.3$
Substitute $b = 0.3$ into [1] or [2] to find a:	From [2]: $a = 0.6 - b = 0.3$ So $a = b = 0.3$.

Finding the variance of a discrete random variable

The variance of X, denoted as $Var(X)$, is a theoretical value that describes how much the values of X vary from the expected value, $E(X)$. Again, if you were to take a sample of observations, you could find the variance of your observations. The more observations you take, the closer your sample variance will be to the theoretical variance, $Var(X)$.

You need to be able to calculate the variance of a discrete random variable.

- $Var(X) = E(X^2) - [E(X)]^2$

You can also use $Var(X) = (x - E(X))^2$

Example 3: The random variable S has the probability distribution:

s	1	2	3	4
$P(S=s)$	$\frac{12}{25}$	$\frac{6}{25}$	$\frac{4}{25}$	$\frac{3}{25}$

Find Var(S).

We start by creating a new row, s^2.	s^2: 4, 1, 1, 4 s: -2, -1, 1, 2 $P(S=s)$: $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{6}$
Find $E(S^2)$.	$E(S^2) = 4\left(\frac{1}{3}\right) + 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) = 2.5$
Find $E(S)$.	$E(S) = -2\left(\frac{1}{3}\right) - 1\left(\frac{1}{3}\right) + 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) = -0.5$
Use the formula $Var(S) = E(S^2) - [E(S)]^2$.	$\therefore Var(S) = 2.5 - (-0.5)^2 = 2.25$

Finding the expected value and variance for a function of X

You might sometimes need to find the expectation of a function of a discrete random variable. Take for example, $E(2X+4)$. The following rules are useful:

- $E(g(X)) = \sum g(x)P(X=x)$

For linear functions, the following rules are much more convenient to use.

- $E(aX+b) = aE(X) + b$ — a and b are constants.
- $E(X+Y) = E(X) + E(Y)$ — X and Y must both be random variables.
- $Var(aX+b) = a^2 Var(X)$

Example 4: A random variable X has $E(X) = 10$ and $Var(X) = 50$. Find:
a) $E(3X)$
b) $E(2X-9)$
c) $Var(4X)$
d) $Var(X-2)$

a) Use $E(aX+b) = aE(X) + b$.	$E(3X) = 3E(X) = 3(10) = 30$
b) Use the same rule as in part a.	$E(2X-9) = 2E(X) - 9 = 2(10) - 9 = 11$
c) Using $Var(aX+b) = a^2Var(X)$	$Var(4X) = 4^2Var(X) = 16(50) = 800$
d) Use the same rule as in part c.	$Var(X-2) = Var(X) = 50$

The random variables S and T are defined as follows: $S = X - 10$ and $T = \frac{1}{2}X - 5$
e) Show that $E(S) = E(T)$.
f) Find $Var(S)$ and $Var(T)$.
A large number of observations of S and T are taken.
g) Comment on any likely differences or similarities.

e) $E(aX+b) = aE(X) + b$	$E(S) = E(X-10) = E(X) - 10 = 10 - 10 = 0$
Use the same rule as above.	$E(T) = E\left(\frac{1}{2}X - 5\right) = \frac{1}{2}E(X) - 5 = \frac{1}{2}(10) - 5 = 5 - 5 = 0$
Form conclusion.	Hence $E(S) = E(T) = 0$.
f) Find both variances using $Var(aX+b) = a^2Var(X)$	$Var(S) = Var(X-10) = Var(X) = 50$ $Var(T) = Var\left(\left(\frac{1}{2}\right)^2 X - 5\right) = \frac{1}{4}Var(X) = \frac{1}{4}(50) = 12.5$
g) Remember that the larger the number of observations, the closer the sample mean and variance will be to $E(X)$ and $Var(X)$	Both means will be close to 0, since this was the expected value for both S and T. Observations for S will be more spread out however since $Var(S) > Var(T)$.

Setting up a probability distribution

Some questions will require you to find the probability distribution of a discrete random variable using context given in the question. When doing so, you need to first find the possible values that the variable can take, then find the associated probabilities. This is the approach used in the next example.

Example 5: Two fair, tetrahedral dice are rolled and D is the difference between their scores. Find the distribution of D.

To find the distribution of D, we need to first figure out which values D can possibly take.	A tetrahedral dice has four sides. The difference between two scores could be 0,1,2 or 3. So D can take the values 0, 1, 2 or 3.
Now we need to find the probability D takes all those possible values. For $P(D=0)$, we multiply by 4 because there are 4 possible ways the score could be the same on both dices. i.e. both could score 1, 2, 3 or 4.	$P(D=0) = P(\text{same score on both dices}) = 4 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{4}$
Similarly, there are two ways this could happen, so we multiply by 2.	$P(D=3) = P(\text{one scores 1, the other 4}) = 2 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}$
Find $P(D=2)$ by similar logic.	$P(D=2) = P(\text{one scores 1, the other 3}) + P(\text{one scores 2, the other 4})$ $= \left(2 \times \frac{1}{4} \times \frac{1}{4}\right) + \left(2 \times \frac{1}{4} \times \frac{1}{4}\right) = \frac{1}{4}$
To find $P(D=1)$, we subtract the other probabilities from 1, since total probability = 1.	$\therefore P(D=1) = 1 - \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = \frac{3}{8}$
So, the probability distribution for D is:	d: 0, 1, 2, 3 $P(D=d)$: $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{4}$, $\frac{1}{8}$

More problems involving random variables

We will now go through some problems that are of a slightly different style to the previous examples and may require you to draw knowledge from other parts of the chapter.

Example 6: The discrete random variable X has a probability distribution given by

x	90°	180°	270°
$P(X=x)$	a	b	0.3

The random variable Y is defined as $Y = sinX°$.
a) Find the range of possible values of $E(Y)$.
b) Given that $E(Y) = 0.2$, write down the values of a and b.

a) Use $E(g(X)) = \sum g(x)P(X=x)$	$E(Y) = E(sinX) = \sum g(x)P(X=x)$ $= \sum sinxP(X=x)$ $= \frac{1}{2}(a) + 0(b) - \frac{1}{2}(0.3) = 0.5a - 0.15$
Since total probability = 1, a must be less than 0.3. Also a is of course a probability so $a \geq 0$.	$By\ inspection, 0 \leq a < 0.7$ $\therefore -0.15 \leq 0.5a - 0.15 < 0.20$ $\Rightarrow -0.15 \leq E(Y) < 0.20$
b) Equate $E(Y)$ to 0.2.	$E(Y) = 0.5a - 0.15 = 0.2$ $0.5a = 0.35 \therefore a = 0.7$
Use the fact that the total probability = 1 to find b.	$a + b + 0.3 = 1$ $\therefore b = 1 - 0.3 - 0.7 = 0$

Example 7: The discrete random variable X has the probability distribution:

x	-2	0	2	3	4
$P(X=x)$	a	b	a	b	c

The random variable Y is defined as $Y = \frac{2-3X}{5}$. Given that $E(Y) = -0.98$ and $P(Y \geq -1) = 0.4$,
a) Write down three simultaneous equations in a, b and c.
b) Solve this system to find the values of a, b and c.

a) First find $E(Y)$ and equate to -0.98.	$E(Y) = -2a + 2a + 3b + 4c = -0.98$
Simplify the equation.	$3b + 4c = -0.98$
Find $P(Y \geq -1)$ and equate to 0.4.	$P(Y \geq -1) = P\left(\frac{2-3X}{5} \geq -1\right) = P(2 - 3X \geq -5) = P(3 \geq 3X)$ $= P(X \leq 1) = P(X=0) + P(X=-2) = a + b$
Simplify the equation.	$\therefore a + b = 0.4$
Using the fact that the total probability is equal to 1:	$a + b + a + b + c = 1$ $2a + 2b + c = 1$
b) Turning our system of equations into a matrix equation and solving using a calculator:	$\begin{pmatrix} 0 & 3 & 4 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -0.98 \\ 0.4 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 3 & 4 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -0.98 \\ 0.4 \\ 1 \end{pmatrix}$
Stating our solutions:	$a = 0.15, b = 0.1, c = 0.5$