Inference – One Sample t-Distribution Cheat Sheet

z-tests vs t-tests

z-scores can be used when testing for whether the mean of a sample population has changed in a hypothesis test, where the z-score is given by:

$$Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Where $\bar{X}$ is the mean found from the sample of size n, μ is the value of the mean given in the null hypothesis and where σ, the population standard deviation, is assumed to be the same as the previously known value.

However, if there is reason to believe that the standard deviation has changed as well, or if the population standard deviation is unknown, it must be estimated from the data. A t-test can be conducted based on the following test statistic:

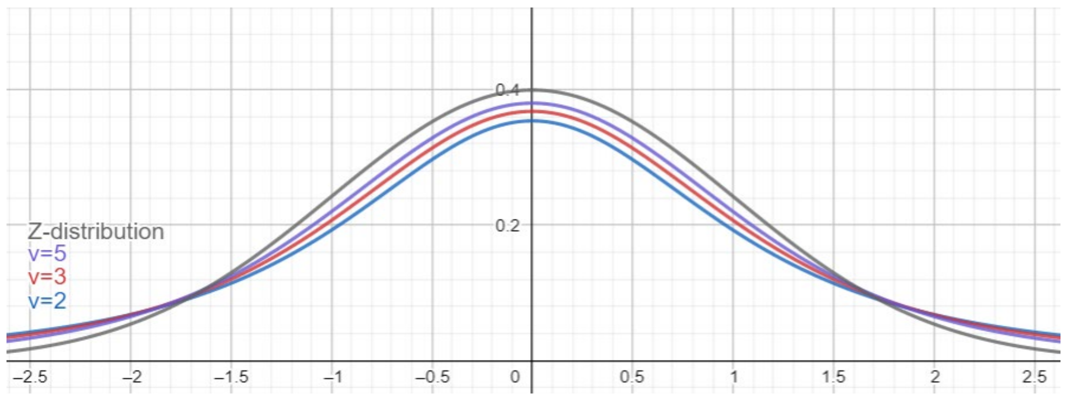
$$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_{n-1}$$

The standard deviation is replaced by the unbiased estimate of the standard deviation. This can be found with the following formula:

$$S^2 = \frac{n}{n-1} \times \sigma^2$$

Where S^2 is the unbiased estimate for the variance. $v = n - 1$ which is the degrees of freedom, which is one less than the sample size, n, because one of the parameters is fixed when you perform a t-test.

In the diagram below, the z- distribution is represented by the grey coloured graph. The other curves represent the t-distribution at different degrees of freedom. As seen from the diagram, the t-distribution approaches the z-distribution as the degrees of freedom become larger.



Example 1: Assuming that X follows a normal distribution and given the following, calculate the test statistic and state the result of the hypothesis test at 5% significance level.

$$H_0: \mu = 73$$

$$H_1: \mu < 73$$

$$n = 6$$

$$\bar{x} = 54$$

$$S = 28.1$$

Calculate the test statistic, T.	$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$ $= \frac{54 - 73}{\frac{28.1}{\sqrt{6}}}$ $= -1.66 (3 s.f.)$
Find the degrees of freedom.	$v = 6 - 1$ $= 5$
Find the critical value from the table given in the formula book when $v = 5$ and $p = 0.95$, since the significance level is 5% and this is a one-tailed test.	Critical value: 2.015
Compare the t score with the critical value and state the conclusion.	$\lvert -1.66 \rvert < 2.015$ ∴ Do not reject H_0. There is insufficient evidence to reject H_0.

Example 2: A manufacturer claims that the bottled orange juices they produce have a volume of 250ml. A random sample of 5 bottles were taken. Given that the mean volume of the sample is 242ml and the variance of the sample is 49.2ml^2, at 10% significance level, test whether the manufacturer's claim is correct and state any assumptions made.

Define the variable and state the assumption needed for this test. A one-sample t-test is used because the population variance is unknown.	Let X be the volume of bottled orange juice in ml. $X \sim N(\mu, \sigma^2)$
State the null and alternative hypotheses.	$H_0: \mu = 250$ $H_1: \mu \neq 250$
Find the unbiased estimate of standard deviation.	$S^2 = \frac{n}{n-1} \times \sigma^2$ $= \frac{5}{5-1} \times 49.2$ $= 61.5$ $S = \sqrt{61.5}$ $= 7.84219 \ldots$
Find the t-score.	$T = \frac{242 - 250}{\frac{7.84219\ldots}{\sqrt{5}}}$ $= -2.28 (3 s.f.)$
Find the degrees of freedom.	$v = 5 - 1$ $= 4$
Find the critical value from the table given in the formula book when $v = 4$ and $p = 0.95$, since the significance level is 10% and this is a two-tailed test.	Critical value: 2.132
Compare the t score with the critical value and state the conclusion.	$\lvert -2.28 \rvert > 2.132$ ∴ Reject H_0. There is sufficient evidence to claim that the manufacturer's claim is inaccurate at a 10% significance level.

Example 3: The average time taken for a team of runners to finish a 5km race is 25.4 minutes. After a month of training, the coach wants to know whether the runners have improved. He took a random sample of 4 runners and recorded their timings, x, in minutes. The following summary statistics were obtained from the sample. Test, at 1% significance level, whether the runners have improved.

$$\sum x = 90, \sum x^2 = 2030$$

State the null and alternative hypotheses.	$H_0: \mu = 25.4$ $H_1: \mu < 25.4$
Find the sample mean.	$\bar{X} = \frac{90}{4}$ $= 22.5$
Find the sample variance.	$\sigma^2 = \frac{\sum x^2}{n} - \bar{x}^2$ $= \frac{2030}{4} - (22.5)^2$ $= 1.25$
Find the unbiased estimate of standard deviation.	$S^2 = \frac{n}{n-1} \times \sigma^2$ $= \frac{4}{4-1} \times 1.25$ $= 1.\dot{6}$ $S = \sqrt{1.\dot{6}}$ $= 1.29099 \ldots$
Find the t-score.	$T = \frac{22.5 - 25.4}{\frac{1.29099\ldots}{\sqrt{4}}}$ $= -4.49 (3 s.f.)$
Find the degrees of freedom.	$v = 4 - 1$ $= 3$
Find the critical value from the table given in the formula book when $v = 3$ and $p = 0.99$, since the significance level is 1% and this is a one-tailed test.	Critical value: 4.541
Compare the t score with the critical value and state the conclusion.	$\lvert -4.49 \rvert < 4.541$ ∴ Accept H_0. There is insufficient evidence to show that the runners have improved.