Discrete Uniform Distribution

A random variable X with a discrete uniform distribution can be written as $X \sim U(n)$. The probability of obtaining each integer from 1 to n is equal and is given by $\frac{1}{n}$:

$$P(X = x) = \frac{1}{n} \qquad x = 1, 2, 3 \dots n$$

A common example of modelling with the discrete uniform distribution is rolling a fair six-sided dice. It can be written as $X \sim U(6)$. X can take the values of any whole number between 1 to 6, and the probability of obtaining each of these is $\frac{1}{6}$.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Modelling with the Discrete Uniform Distribution

Sometimes a random variable has equal chance of taking any integers within a range, but the range does not begin with 1, or the random variable can only take certain integers in the range. In this case, the random variable can be modelled using linear transformation.

Example 1: A random variable Y can take any even integer ranging from 10 to 30 . The probability of Y taking each of these values is equal. Show that Y can be written in the form $aX + b$ and find the value of n for $X \sim U(n)$.

Write Y as a linear transformation of X.	$Y = aX + b$
Since Y can only take even numbers, $a = 2$.	$Y = 2X + b$
When $X = 1, Y = 10$.	$10 = 2(1) + b$ $b = 8$ $\Rightarrow Y = 2X + 8$
Find n such that $Y = 30$ when $x = n$.	$30 = 2n + 8$ $2n = 22 \Rightarrow n = 11$

Mean of the Discrete Uniform Distribution

For a random variable X with a discrete uniform distribution $X \sim U(n)$, the mean is given by:

$$E(X) = \frac{n+1}{2}$$

Example 2: The random variable X has a discrete uniform distribution $X \sim U(n)$, show that $E(X) = \frac{n+1}{2}$.

Find $E(X)$ by multiplying each possible value of X by the corresponding probability, then adding them up together. For $X \sim U(n)$, X can be any integer from 1 up to n, and the probability for each is $\frac{1}{n}$.	$E(X) = \sum x_i p_i$ $= 1 \times \frac{1}{n} + 2 \times \frac{1}{n} + 3 \times \frac{1}{n} + \cdots + n \times \frac{1}{n}$
Factorise out $\frac{1}{n}$.	$= \frac{1}{n} \times (1 + 2 + 3 \dots \dots + n)$
Rewrite $(1 + 2 + 3 + \cdots + n)$ as $\sum_{r=1}^{n} r$.	$= \frac{1}{n} \sum_{r=1}^{n} r$
Use the result for the sum of the first n positive integers $\left(\sum_{r=1}^{n} r = \frac{n(n+1)}{2}\right)$ to simplify the formula and end the proof.	$= \frac{1}{n} \times \frac{n(n+1)}{2}$ $= \frac{n+1}{2}$

Variance of the Discrete Uniform Distribution

For a random variable X with a discrete uniform distribution $X \sim U(n)$, the variance is given by:

$$Var(X) = \frac{n^2 - 1}{12}$$

Example 3: The random variable X has a discrete uniform distribution $X \sim U(n)$. Show that $Var(X) = \frac{n^2-1}{12}$.

Find $E(X^2)$.	$E(X^2) = \sum x_i^2 p_i$ $= 1^2 \times \frac{1}{n} + 2^2 \times \frac{1}{n} + 3^2 \times \frac{1}{n} + \cdots + n^2 \times \frac{1}{n}$ $= \frac{1}{n} \times (1^2 + 2^2 + 3^2 \dots \dots + n^2)$ $= \frac{1}{n} \times \sum_{r=1}^{n} r^2$
Use the result $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.	$= \frac{1}{n} \times \frac{n}{6}(n+1)(2n+1)$ $= \frac{1}{6}(n+1)(2n+1)$
Find $E(X)$. (Refer to Example 2.)	$E(X) = \frac{n+1}{2}$
Use the formula of variance.	$Var(X) = E(X^2) - (E(x))^2$ $= \frac{1}{6}(n+1)(2n+1) - \left(\frac{n+1}{2}\right)^2$ $= \frac{1}{6}(n+1)(2n+1) - \frac{1}{4}(n+1)(n+1)$ $= (n+1)\left(\frac{2n+1}{6} - \frac{n+1}{4}\right)$ $= (n+1)\left(\frac{4n+2-3n-3}{12}\right)$ $= (n+1)\left(\frac{n-1}{12}\right)$ $= \frac{n^2-1}{12}$

Example 3: A random variable X has the discrete uniform distribution $X \sim U(8)$. Find its mean and variance.

Use the formula $E(X) = \frac{n+1}{2}$ to find $E(X)$ when $n = 8$.	$E(X) = \frac{8+1}{2}$ $= \frac{9}{2}$
Use the formula $Var(X) = \frac{n^2-1}{12}$ to find $Var(X)$ when $n = 8$.	$Var(X) = \frac{8^2-1}{12}$ $= \frac{63}{12}$