Discrete Random Variables and Expectation I Cheat Sheet

AQA A Level Further Maths: Statistics

Probability Distributions for a Discrete Random Variable

Discrete random variables are variables that can only take distinct values; they can change each time they are measured. The most common example is the outcome of rolling a die.

The probability distribution of a discrete random variable, X, can be shown by a table or a function. Here is an example of the same probability distribution displayed as a table and as a function. $P(X = x)$ is defined as the probability of X (e.g. the outcome of rolling a die) taking the value of x.

x	1	2	3	4
$P(X = x)$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

$$P(X = x) = \frac{x}{10} \qquad x = 1, 2, 3, 4$$

Measures of Average and Spread for a Discrete Random Variable

Mean and Expectation

Expectation is denoted by $E(X)$ and is a measure of average for a random variable. It represents the outcome expected when the random variable is measured an infinite of times.

For discrete random variables, the expectation can be calculated by multiplying each possible value the random variable can take (x_i) by the probability of obtaining it (p_i) and then adding them all together:

$$E(X) = \sum x_i p_i$$

Example 1: A random variable X has the probability distribution below. Find $E(X)$.

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{20}$	$\frac{1}{5}$

Multiply each possible value of X by the corresponding probability, then sum them up together.	$E(X) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{5}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{1}{20}\right) + 5\left(\frac{1}{5}\right)$ $= \frac{1}{4} + \frac{2}{5} + \frac{9}{10} + \frac{1}{5} + 1$ $= 2.75$

Variance

The mean of the squares of the values that the random variable can take is denoted by $E(X^2)$ and is calculated using the formula:

$$E(X^2) = \sum x_i^2 p_i$$

Variance is a measure of how spread out a set of data is. The variance of a discrete random variable is given by the formula:

$$\sigma^2 = Var(X) = E(X^2) - (E(X))^2$$

In words, the variance is the mean of the squares minus the square of the mean. This gives a measurement of how far away the values in a set of data are from the mean.

Standard Deviation

Standard deviation is another measure of spread and is found by taking the positive square root of variance.

$$\sigma = \sqrt{Var(X)}$$

Example 2: The random variable X has the probability distribution shown in the table. Given that $E(X) = 4$, find the variance and standard deviation of X.

x	0	2	4	5	q
$P(X = x)$	$\frac{1}{10}$	p	$\frac{1}{4}$	$2p$	$\frac{1}{5}$

Find the value of P. Remember that the probabilities should add up to 1.	$\frac{1}{10} + p + \frac{1}{4} + 2p + \frac{1}{5} = 1$ $3p = 1 - 0.55$ $p = 0.15$
Find the value of q using that $E(X) = 4$.	$0(0.1) + 2(0.15) + 4(0.25) + 5(0.3) + q(0.2) = 4$ $0.2q = 4 - 2.8$ $q = 6$
Find $E(X^2)$.	(table below) $E(X^2) = 0 + 0.6 + 4 + 7.5 + 7.2$ $= 19.3$
Calculate the variance using $E(X^2) - (E(X))^2$.	$19.3 - 4^2 = 19.3 - 16$ $= 3.3$
Calculate the standard deviation as the positive square root of the variance.	$\sigma^2 = \sqrt{3.3}$ $= 1.82$ (3s.f.)

x	0	2	4	5	6
$P(X = x)$	0.1	0.15	0.25	0.3	0.2
$x_i^2 p_i$	0	$2^2(0.15) = 0.6$	$4^2(0.25) = 4$	$5^2(0.3) = 7.5$	$6^2(0.2) = 7.2$

Median and Mode

Median and mode are two measures of average that are less frequently used.

Median is a value for which $P(X \leq M) \leq 0.5$ and $P(X \geq M) \geq 0.5$. Note that the value of M does not have to be a value that the random variable X can take - when there are two values that fulfil the above criteria, their mean is taken to be the median.

The mode of the random variable X is the value of x for which $P(X = x)$ is the highest.

Example 3: The random variable X has the probability distribution shown in the table. State the median and the mode.

x	3	7	8	10	11
$P(X = x)$	0.2	0.2	0.1	0.25	0.25

Find the value of M for which $P(X \leq M) \leq 0.5$ and $P(X \geq 5) \geq 0.5$.	$P(X \leq 8) = 0.5$ $P(X \geq 8) = 0.6$ $\therefore M = 8$
Find the value of x associated with the highest probability. There are two values associated with the highest probability here, so there are two modes.	10 and 11

Functions of Discrete Random Variables

Expectation and Variance of a Linear Function of a Discrete Random Variable

When a variable X is transformed through a linear function, such as multiplication by a constant or adding a constant, the expectation and variance of this variable can be found using the expectation and variance of X. For the new variable $Y = aX + b$, where a and b are some constants,

$$E(Y) = aE(X) + b$$

$$Var(Y) = a^2 Var(X)$$

Example 4: The random variable X is transformed into variable Y by $Y = 64 - 3X$. Given that $E(X) = 7$ and $Var(X) = 12$, find the expectation and variance of Y.

Find $E(Y)$ using $E(Y) = aE(X) + b$.	$E(Y) = (-3)(7) + 64$ $= 43$
Find $Var(Y)$ using $Var(Y) = a^2 Var(X)$.	$Var(Y) = (-3)^2(12)$ $= 108$

Expectation of a General Function of a Discrete Random Variable (A Level Only)

When a function g is applied to the random variable X, the new expectation can be found using the following:

$$E(g(X)) = \sum g(x_i) p_i$$

Example 5: The random variable X has the following probability density distribution:

x	0	30	60	90
$P(X = x)$	0.25	0.35	0.15	0.25

Given that $Y = \cos(x)$, find $E(Y)$.

Find $E(Y)$ using $E(g(X)) = \sum g(x_i) p_i$.	$E(Y) = E(\cos x)$ $= \sum \cos(x_i) p_i$ $= 0.25 \cos 0 + 0.35 \cos 30 + 0.15 \cos 60 + 0.25 \cos 90$ $= 0.25(1) + 0.35\left(\frac{\sqrt{3}}{2}\right) + 0.15\left(\frac{1}{2}\right) + 0.25(0)$ $= 0.628$ (3 s.f.)

Mean, Variance and Standard Deviation of Non-linear Functions of a Discrete Random Variable (A Level Only)

When a variable X is transformed by a non-linear function, the new expectation and variance cannot be found directly. The probability distribution of the new variable needs to be found first

Example 6: The random variable X has the following probability density distribution:

$$P(X = x) = \frac{x}{10}(4 - x) \qquad x = 1, 2, 3.$$

The random variable Y is given as $Y = X^2$. Find the mean, variance, and standard deviation for Y.

Construct the probability distribution table for both X and Y. Find the value of Y using $Y = X^2$. For each value of y calculated from a value of x, $P(Y = y)$ is equivalent to $P(X = x)$.	(table below)
Find $E(Y)$.	$E(Y) = 0.3(1) + 0.4(4) + 0.3(9)$ $= 0.3 + 1.6 + 2.7$ $= 4.6$
Find $E(Y^2)$.	$E(Y^2) = 0.3(1)^2 + 0.4(4)^2 + 0.3(9)^2$ $= 0.3 + 6.4 + 24.3$ $= 31$
Find $Var(Y)$ using $E(Y^2)$ and $E(Y)$.	$Var(Y) = E(Y^2) - (E(Y))^2$ $= 31 - 4.6^2$ $= 31 - 21.16$ $= 9.84$
Find the standard deviation by taking the positive square root of the variance.	$\sigma = \sqrt{9.84}$ $= 3.14$ (3s.f.)

x	1	2	3
y	$1^2 = 1$	$2^2 = 4$	$3^2 = 9$
P	$\frac{1}{10}(4-1) = 0.3$	$\frac{2}{10}(4-2) = 0.4$	$\frac{3}{10}(4-3) = 0.3$