Mean, Variance and Standard Deviation of a Given Probability Density Function

Expectation of a Continuous Random Variable

The mean, or the expectation of a continuous variable X, which is defined on the domain $a < x < b$, is given by:

$$E(X) = \int_a^b x\, f(x)\, dx$$

Expectation of the Square of a Continuous Random Variable

The expectation of the square of a continuous variable X which is defined for the domain $a < x < b$, is given by:

$$E(X^2) = \int_a^b x^2\, f(x)\, dx$$

Variance of a Continuous Random Variable

The variance of a continuous random variable can be calculated by:

$$Var(X) = E(X^2) - (E(X))^2$$

The standard deviation can be obtained by square rooting the variance.

Example 1: The continuous random variable X has the probability density function:

$$f(x) = \begin{cases} \frac{3}{20}(4x^2 - x^3) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the standard deviation of X.

Find $E(X)$.	$E(X) = \int_0^2 x \times \frac{3}{20}(4x^2 - x^3)\, dx$ $= \frac{3}{20}\int_0^2 4x^3 - x^4\, dx$ $= \frac{3}{20}\left[x^4 - \frac{x^5}{5}\right]_1^2$ $= \frac{3}{20}\left[\left(16 - \frac{32}{5}\right) - \left(1 - \frac{1}{5}\right)\right]$ $= \frac{3}{20}\left(\frac{48}{5} - \frac{4}{5}\right)$ $= 1.32$
Find $E(X^2)$.	$E(X^2) = \int_0^2 x^2 \times \frac{3}{20}(4x^2 - x^3)\, dx$ $= \frac{3}{20}\int_0^2 4x^4 - x^5\, dx$ $= \frac{3}{20}\left[\frac{4x^5}{5} - \frac{x^6}{6}\right]_1^2$ $= \frac{3}{20}\left[\left(\frac{128}{5} - \frac{64}{6}\right) - \left(\frac{4}{5} - \frac{1}{6}\right)\right]$ $= \frac{3}{20}\left(\frac{224}{15} - \frac{19}{30}\right)$ $= 2.145$
Calculate the variance using $E(X^2) - (E(X))^2$.	$Var(X) = 2.145 - (1.32)^2$ $= 0.4026$
Calculate the standard deviation as the square root of variance.	$\sigma = \sqrt{0.4026}$ $= 0.635$ (3s.f.)

Functions of a Continuous Random Variable

Expectation and Variance of a Linear Function of a Continuous Random Variable

When a variable X undergoes a linear transformation, such as multiplication by the constant a, or adding a constant b, the expectation and variance of the new variable can be found in terms of $E(X)$ and $Var(X)$.

$$E(aX + b) = aE(X) + b$$

$$Var(aX + b) = a^2 Var(X)$$

Example 2: The continuous random variable X has the expectation and variance:

$$E(X) = 1.24$$

$$Var(X) = 1.86$$

Given that Y is a continuous random variable such that $Y = 3X - 2$, find:

a) $E(Y)$

b) $Var(Y)$

c) $E(Y^2)$

d) the standard deviation of Y

a) Find $E(Y)$ from $E(X)$.	$E(Y) = 3E(X) - 2$ $= 3(1.24) - 2$ $= 1.72$
b) Find $Var(Y)$ from $Var(X)$.	$Var(Y) = 3^2(1.86)$ $= 16.74$
c) Find $E(Y^2)$ using $Var(Y) = E(Y^2) - (E(Y))^2$.	$16.74 = E(Y^2) - (1.72)^2$ $E(Y^2) = 16.74 - 2.9584$ $= 13.7816$
d) Calculate the standard deviation from $Var(Y)$.	$\sigma = \sqrt{16.74}$ $= 4.09$ (3s.f.)

Expectation of a General Function of a Continuous Random Variable

The expectation of any general function $g(x)$ for a continuous random variable X with the probability density function $f(x)$ can be found by:

$$E(g(x)) = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Example 3: The continuous random variable X has the probability density function:

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $E\left(\frac{4}{X}\right)$.

Find $E\left(\frac{4}{x}\right)$ using the formula above.	$E\left(\frac{4}{X}\right) = \int_0^1 \frac{4}{x}\, f(x)\, dx$ $= \int_0^1 \frac{4}{x}(3x^2)\, dx$ $= \int_0^1 12x\, dx$ $= \left[\frac{12x^2}{2}\right]_0^1 = 6$

Mean, Variance and Standard Deviation of Non-Linear Functions of a Continuous Random Variable

By using the formula for the expectation of a general function of a continuous random variable, the variance and standard deviation can also be found for non-linear functions of a continuous random variable.

Example 4: The continuous random variable X has the probability density function:

$$f(x) = \begin{cases} \frac{3x}{4}(2 - x) & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Find $Var(X^2)$ and the standard deviation.

Let $Y = X^2$ so $Var(X^2) = Var(Y) = E(Y^2) - (E(Y))^2$. Find $E(Y)$.	$E(Y) = E(X^2)$ $= \int_0^2 x^2 \times \frac{3x}{4}(2 - x)\, dx$ $= \frac{3}{4}\int_0^2 2x^3 - x^4\, dx$ $= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$ $= \frac{3}{4}\left(\frac{32}{4} - \frac{32}{5} - 0\right)$ $= 1.2$
Find $E(Y^2)$.	$E(Y^2) = E((X^2)^2)$ $= E(X^4)$ $= \int_0^2 x^4 \times \frac{3x}{4}(2 - x)\, dx$ $= \frac{3}{4}\int_0^2 2x^5 - x^6\, dx$ $= \frac{3}{4}\left[\frac{2x^6}{6} - \frac{x^7}{7}\right]_0^2$ $= \frac{3}{4}\left(\frac{128}{6} - \frac{128}{7} - 0\right)$ $= \frac{16}{7}$
Find $Var(Y)$.	$Var(Y) = E(Y^2) - (E(Y))^2$ $= \frac{16}{7} - (1.2)^2$ $= 0.846$ (3s.f.) $Var(X^2) = 0.846$ (3s.f.)
Find the standard deviation from $Var(Y)$.	$\sigma = \sqrt{0.84571\ldots}$ $= 0.920$ (3s.f.)