Motion in 1D with a Variable Applied Force

Newton's Law with a Variable Force

When a variable force is applied (and the force is given as a function of displacement/time/velocity), Newton's second law ($F = ma$) can be applied by expressing acceleration as a function of displacement/time/velocity as well.

F is expressed as	Form of acceleration to use	Equation formed by applying $F = ma$
A function of displacement, x	$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$	$\frac{1}{2}mv^2 = \int F\ dx\ (+c)$
A function of time, t	$a = \frac{dv}{dt}$	$mv = \int F\ dt\ (+c)$
A function of velocity, v	$a = \frac{dv}{dt}$ or $a = v\frac{dv}{dx}$	$t = \int \frac{m}{F}\ dv$ or $x = \int \frac{mv}{F}\ dv$

Example 1: A single variable force $F = (25 - v^2)$N is acted on a particle at rest at a point O. The particle then moves in the direction of the force applied. Given that the particle has a mass of 4kg, find the displacement from O when $v = 3\text{ms}^{-1}$.

Substitute $a = v\frac{dv}{dx}$ into $F = ma$.	$25 - v^2 = 4v\frac{dv}{dx}$
Separate the variables.	$dx = \frac{4v}{25 - v^2}\ dv$
Integrate both sides.	$\int dx = \int \frac{4v}{25 - v^2}\ dv$ $x = -2\ln\lvert 25 - v^2\rvert + c$
Find c.	When $v = 0, x = 0$, $\therefore c = 2\ln(25)$
Substitute $v = 3$ into the equation.	$x = -2\ln\lvert 25 - 3^2\rvert + 2\ln(25)$ $= 0.893\text{m}\ (3\text{ s.f.})$

Newton's Law of Gravitation (Inverse Square Law)

Newton's law of gravitation states that for two bodies with masses M_1 and M_2, the force of gravitational attraction is directly proportional to the product of their masses and inversely proportional to the squared value of the distance between their centres (d^2). This is given by the formula:

$$F = \frac{GM_1M_2}{d^2}$$

, where G is the constant of gravitation $\approx 6.67x10^{-11}\ \text{Nm}^2\text{kg}^{-2}$.

Example 2: A particle with mass $m$kg is at a distance of $(x - R)$m above the surface of the Earth, which is modelled as a sphere with radius Rm, where $x \geq R$. Given that the acceleration due to gravity on the surface of the Earth is $g\text{ms}^{-2}$, and that the magnitude of gravitational force is inversely proportional to the distance between the particle and the centre of Earth, show that the gravitational force acting on the particle is $\left(\frac{mgR^2}{x^2}\right)$N.

First, write gravitational force in the form of $F = \frac{k}{x^2}$ as we are given this proportionality in the question. Find k by using $F = mg$ on the surface of Earth (R above its centre).	$F = \frac{k}{x^2}$ $mg = \frac{k}{R^2}$ $k = mgR^2$ $\therefore F = \frac{mgR^2}{x^2}$

Simple Harmonic Motion

For a particle P undergoing simple harmonic motion (S.H.M.), its acceleration is always towards a fixed point O, which is on the line of motion of P. The magnitude of this acceleration is directly proportional to x, the displacement of P from O. This is given by:

$$\ddot{x} = -\omega^2 x$$

where ω is the angular velocity measured in rads^{-1}, $\dot{x} = \frac{dx}{dt} = v$ and $\ddot{x} = \frac{d^2x}{dt^2} = a$.

The velocity of the particle can be found using the following, where a is the amplitude of motion and $-a \leq x \leq a$ for all values of t,

$$v^2 = \omega^2(a^2 - x^2)$$

The following equations can be used to find displacement given different starting positions of P at $t = 0$:

Position of P at $t = 0$	Equation for displacement
Centre of oscillation	$x = a\sin\omega t$
End point of oscillation	$x = a\cos\omega t$
Other points	$x = a\sin(\omega t + \alpha)$, where $t = \frac{-\alpha}{\omega}$ when $x = 0$

The period of oscillation is given by:

$$T = \frac{2\pi}{\omega}$$

Example 3: Particle P has a mass of 0.8kg and moves along a straight line in the direction OP. The distance between P and the fixed point O is x metres at t seconds. A resistive force of $F = 16x$ N acts on P in the direction PO. a.) Show that P is moving with S.H.M. and find the value of ω. b.) Given that the amplitude of the S.H.M. is 1.2m , find the displacement at $t = 6$.

a.) Taking the direction OP as positive, apply Newton's second law and write the equation in the form of $\ddot{x} = -\omega^2 x$. Since the force is resistive, it is negative.	$F = ma$ $-16x = 0.8(\ddot{x})$ $\Rightarrow \ddot{x} = -20x$ $a \propto -x\ \therefore$ particle P is moving with S.H.M. $\omega = \sqrt{20} = 4.47\text{rads}^{-1}$ (3s.f.)
b.) Take $t = 0$ when P is at the centre of oscillation.	$x = a\sin\omega t$ $= 1.2\sin(\sqrt{20} \times 6)$ $= 1.19\text{m}(3\text{ s.f.})$

S.H.M. of a Particle Attached to a Horizontal Elastic Spring or String

For a particle P attached to one end of an elastic spring and moving on a smooth horizontal surface, P will have S.H.M. with complete oscillations since there is a constant presence of a restoring force.

If P is attached to an elastic string instead, it will have S.H.M. only when the string is taut (when there is tension) and moves with constant speed when the string is slack (absence of tension).

Example 4: Particle P has a mass of 0.5kg and is attached to one end of an elastic string, which has another end attached to a fixed point O on the horizontal plane. The natural length of the string is 0.4m and the modulus of elasticity is 8N. P is pulled away horizontally from O so that $OP = 0.6$m before it is released. a.) Find the speed of P when the string returns to its natural length. b.) Find the period of the S.H.M. of P when the string is taut. c.) Find the time taken for P to first return to its starting position.

a.) Find the tension in the string in terms of x using Hooke's law.	$T = \frac{\lambda x}{l} = \frac{8x}{0.4} = 20x\text{N}$
Using $F = ma$ and $\ddot{x} = -\omega^2 x$, find ω^2.	$20x = -0.5\ddot{x} \Rightarrow \ddot{x} = -40x$ $\Rightarrow \omega^2 = 40$
Find v using $v^2 = \omega^2(a^2 - x^2)$.The initial extension is the amplitude. At natural length, $x = 0$.	$v^2 = 40(0.2^2 - 0^2) = 1.6$ $\Rightarrow v = \sqrt{1.6} = 1.26\ \text{ms}^{-1}(3\text{ s.f.})$
b.) Find the period of oscillation using $T = \frac{2\pi}{\omega}$.	$T = \frac{2\pi}{\sqrt{40}} = 0.993\text{s}\ (3\text{ s.f.})$
c.) Find the time interval for when the string is slack using $t = \frac{d}{v}$. The string remains slack when it is within its natural length in both directions.	$t = \frac{2 \times 0.4}{\sqrt{1.6}}$ $= 0.63245 \ldots$
For P to return to its starting position, it must have travelled a total of twice the intervals when the string is slack and a complete oscillation.	Total time taken $= 0.99345 \ldots + (2 \times 0.63245)$ $= 2.26\text{s}\ (3\text{ s.f.})$

S.H.M. of a Particle Attached to a Vertical Elastic Spring or String

When P is attached to one end of an elastic spring vertically and displaced downwards from its equilibrium, P moves with S.H.M. with complete oscillation, with the equilibrium position as the centre of oscillation.

When it is attached to an elastic string instead, it moves with S.H.M. when the string is taut and has complete oscillations when the amplitude is less than the equilibrium extension. When the amplitude exceeds equilibrium extension, the string becomes slack and P moves freely under gravity.

Example 5: A particle P has mass 0.5kg and is attached to one end of a light elastic string. The other end of the string is attached to a fixed point O on the ceiling. The string has a natural length of 0.4m and a modulus of elasticity of 5N. a.) Find the extension of the string in terms of g when the particle is hanging in equilibrium. b.) P is released from rest from point O. Find the length of the string when it first comes to an instantaneous rest using conservation of energy. c.) Find the time interval when P is falling freely under gravity before it first becomes taut.

a.) At equilibrium, the weight of the particle is equal to the tension in the string. Find the extension of the string e using Hooke's law.	$mg = \frac{\lambda x}{l}$ $0.5g = \frac{5e}{0.4} \Rightarrow e = 0.04gm$
b.) At both points, $v = 0$ so $KE = 0$.	Change in KE $= 0$
Calculate the gravitational potential energy lost.	Let length of string when P first comes to an instantaneous rest$= h$ GPE lost $= mgh = (0.5)(9.8)h$ $= 4.9h\text{J}$
Calculate the elastic potential energy gained.	EPE gained $= \frac{\lambda x^2}{2l}$ $= \frac{5(h - 0.4)^2}{2 \times 0.4}$
Using the law of conservation of energy, find h.	Gain in EPE $=$ Loss in GPE $\frac{5(h - 0.4)^2}{2 \times 0.4} = 4.9h$ $\Rightarrow 5h^2 - 4h + 0.8 = 3.92h$ $\Rightarrow 5h^2 - 7.92h + 0.8 = 0$ $h = 0.108$ or $h = 1.48$ $h = 1.48\text{ m}(3\text{ s.f.})$, since h should be greater than the natural length of the string.
c.) The string moves freely under gravity so the SUVAT equations can be used.	$s = ut + \frac{1}{2}at^2$ $0.4 = 0(t) + \frac{1}{2}(9.8)t^2 \Rightarrow t^2 = \frac{0.4}{4.9}$ $\Rightarrow t = \sqrt{\frac{0.4}{4.9}} = 0.286\text{s}\ (3\text{ s.f.})$