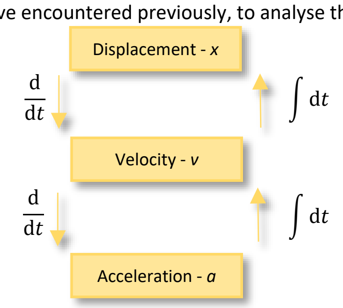
Kinematics Cheat Sheet

Edexcel FM2

In this chapter, you will learn how to use calculus to deal with problems where acceleration isn't constant, and instead varies with time, displacement, or velocity. You will become familiar with the different ways we can model real-world situations, such as drag or air resistance.

Acceleration varying with time

We can use the relationship between time, displacement, velocity, and acceleration, along with calculus techniques we've encountered previously, to analyse the motion of a particle whose acceleration varies with time.



- We can differentiate displacement with respect to time to obtain velocity: $v = \frac{dx}{dt}$
- We can differentiate velocity with respect to time to obtain acceleration: $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$
- Similarly, we can integrate acceleration with respect to time to obtain velocity: $v = \int a \, \mathrm{d}t$
- We can integrate velocity with respect to time to obtain displacement: $x = \int v \, \mathrm{d}t$

Don't forget to include your constant of integration when finding these quantities. We can use information given in the question, for example initial conditions, to find the value of this constant.

Example 1: A particle P travels along the x-axis. It starts from rest at the origin O and at a time t seconds after starting, P's velocity is v m/s, where $v = 5\sin\left(\frac{\pi}{2}t\right)$. Find a) the acceleration of P 1.5 minutes after it starts moving, and b) the greatest displacement from O it can reach while moving.

Differentiate the velocity with respect to time to find the acceleration, a ms^{-2}.	$a = \frac{dv}{dt} = \frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)$
Convert the value of t into seconds and sub into value for a.	1.5 minutes $= 90$ seconds so $a = \frac{5\pi}{2}\cos\left(\frac{\pi}{2} \times 90\right) = \frac{5\pi}{2}\cos(45\pi) = \frac{5\pi}{2} \times -1 = \frac{-5\pi}{2}$ P accelerates at $\frac{5\pi}{2}$ ms^{-2} in the negative x direction
Integrate the velocity with respect to time to find the displacement, xm, including the integration constant in the expression.	$x = \int 5\sin\left(\frac{\pi}{2}t\right)\mathrm{d}t = -\frac{2}{\pi} \times 5\cos\left(\frac{\pi}{2}t\right) + c = -\frac{10}{\pi}\cos\left(\frac{\pi}{2}t\right) + c$
Use the initial condition to find the value of c.	At t=0, x=0: $-\frac{10}{\pi}\cos(0) + c = 0 \rightarrow c = \frac{10}{\pi}$ $\therefore x = \frac{10}{\pi}\left(1 - \cos\left(\frac{\pi}{2}t\right)\right)$
Use knowledge of the cosine function to find the maximum displacement.	$\cos\left(\frac{\pi}{2}t\right)$ varies between -1 and 1, so x is greatest when $\cos\left(\frac{\pi}{2}t\right) = -1$ (subtracting the most negative value). $\therefore$ greatest displacement $= \frac{10}{\pi}(1 - (-1)) = \frac{20}{\pi}$ The greatest distance from O that P reaches is $\frac{20}{\pi}$ m.

Acceleration varying with displacement

When a particle moves in a straight line with acceleration that varies with displacement, we use calculus to manipulate the relationships we used above to analyse its motion. We can link acceleration, velocity, and displacement when considering differentiation with respect to time:

- $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

Now, we use the **chain rule** to get these relationships in terms of a change in displacement:

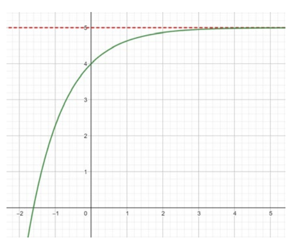
- $a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = v \cdot \frac{dv}{dx}$ as $v = \frac{dx}{dt}$

We compare this with the result obtained when we implicitly differentiate $\frac{1}{2}v^2$:

- $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \times 2v \times \frac{dv}{dx} = v \cdot \frac{dv}{dx}$

So, we conclude

- $a = v \cdot \frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$



The **terminal** or **limiting velocity** of an object is the velocity it approaches but can't exceed, represented by an asymptote on its velocity-displacement graph.

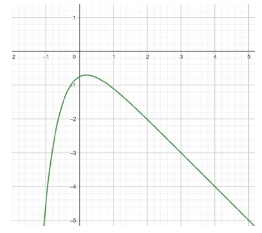
The terminal velocity for $v = 5 - e^{-x}$ is $v = 5$.

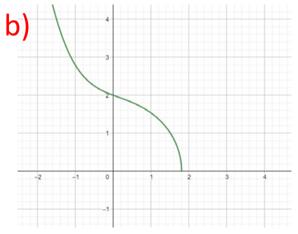
We can use these different expressions when we have a particle whose acceleration varies with displacement.

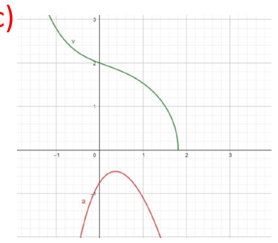
- e.g. if we know $a = f(x)$, we can rewrite this as $f(x) = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, or integrate to show $\frac{1}{2}v^2 = \int f(x)\,\mathrm{d}x$

Example 2: Model a robot as a particle P moving along the x-axis. When it passes through the origin O, it has a velocity 2 ms^{-1} along x. P is x m from O at a time t seconds, with a velocity of v ms^{-1} and an acceleration of magnitude $(\frac{3}{4}e^{-2x} + x)$ ms^{-2} towards O. Find v in terms of x for $x > 0$ and explain why x has a limited range of values.

Use the relation between acceleration and velocity, solving the differential equation to find v. Note that acceleration is in the direction of x decreasing.	$a = -\left(\frac{3}{4}e^{-2x} + x\right) \rightarrow \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{3}{4}e^{-2x} - x$ $\frac{1}{2}v^2 = +\frac{3}{8}e^{-2x} - \frac{1}{2}x^2 + A$
Find the integration constant using the initial condition given in the question.	At $x = 0, v = 2 \therefore \frac{1}{2}(2^2) = \frac{3}{8} + A \rightarrow A = 2 - \frac{3}{8} = \frac{13}{8}$
Rearrange to make v the subject.	$\frac{1}{2}v^2 = \frac{3}{8}e^{-2x} - \frac{1}{2}x^2 + \frac{13}{8} \rightarrow v = \sqrt{\frac{3}{4}e^{-2x} - x^2 + \frac{13}{4}}$
Use the fact that the argument in the square root can't be negative to explain why x can't take every value above 0. You can't solve this inequality, but the question doesn't require you to.	$\frac{3}{4}e^{-2x} + \frac{13}{4}$ is always > 0, so require $\frac{3}{4}e^{-2x} + \frac{13}{4} > x^2$

a) 

b) 

c) 

The physical interpretation for the limited range of x values in graph b is that P comes to a stop after decelerating.

a) Acceleration-displacement for example 2
b) Velocity-displacement for example 2
c) Acceleration & velocity on the same axes for example 2

Acceleration varying with velocity

Sometimes, acceleration is given as a function of velocity. In this case, we can form and solve a differential equation to acquire velocity as a function of time.

We know

- $a = \frac{dv}{dt}$

and can use this relation to set up a differential equation of the form

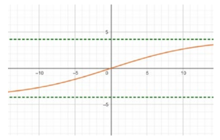
- $\frac{dv}{dt} = f(v)$.

When an object is at its **terminal** or **limiting speed**, its acceleration is **zero.**

We then solve this using the separation of variables technique taught in Pure Year 2, Section 11.10.
Similarly, if we need to find the distance travelled when acceleration varies with velocity, we use

- $a = v \cdot \frac{dv}{dx}$

Example 3: A cyclist moves along a straight horizontal track. The cyclist's acceleration is $\frac{16-v^2}{50}$ ms^{-2} at a time t seconds, where v ms^{-1} is their velocity. Given they start from rest at the origin, find v in terms of t and show that the cyclist won't exceed a velocity of $4ms^{-1}$. Find the distance covered by the cyclist as they accelerate from $0ms^{-1}$ to $2ms^{-1}$, $x, v > 0$ for all t.

Using $a = \frac{dv}{dt}$, separate the variables to set up a differential equation.	$\frac{dv}{dt} = \frac{16 - v^2}{50} \rightarrow \frac{1}{16 - v^2}\frac{dv}{dt} = \frac{1}{50} \rightarrow \int \frac{1}{16 - v^2}\mathrm{d}v = \int \frac{1}{50}\mathrm{d}t$
Use the method of partial fractions to integrate the LHS. We manipulate our expression to make it easier to work with later on.	$16 - v^2 = (4 + v)(4 - v)$, so let $\frac{1}{16-v^2} = \frac{A}{4+v} + \frac{B}{4-v}$ Multiply through by $(4 + v)(4 - v)$ to give $1 = A(4 - v) + B(4 + v)$ When $v = 4: 1 = 8B \rightarrow B = \frac{1}{8}$ When $v = -4: 1 = 8A \rightarrow A = \frac{1}{8}$ Thus, we have $\frac{1}{8}\int \frac{1}{4+v}\mathrm{d}v + \frac{1}{8}\int \frac{1}{4-v}\mathrm{d}v = \int \frac{1}{50}\mathrm{d}t \therefore \frac{1}{8}(\ln(4 + v) - \ln(4 - v)) = \frac{t}{50} + C$ $\ln\left(\frac{4 + v}{4 - v}\right) = 8\left(\frac{t}{50} + C\right) \rightarrow \left(\frac{4 + v}{4 - v}\right) = e^{\left(\frac{4}{25}t + 8C\right)}$ e^{8C} is a constant, so $e^{\left(\frac{4}{25}t + 8C\right)} = D \cdot e^{\left(\frac{4}{25}t\right)} \rightarrow \left(\frac{4 + v}{4 - v}\right) = D \cdot e^{\left(\frac{4}{25}t\right)}$
We find the integration constant from the initial conditions. Now, we can make velocity the subject.	When $t = 0, v = 0 \therefore \frac{4}{4} = De^0 \rightarrow D = 1$ $\left(\frac{4 + v}{4 - v}\right) = e^{\left(\frac{4}{25}t\right)} \rightarrow 4 + v = 4e^{\frac{4}{25}t} - ve^{\frac{4}{25}t} \rightarrow v\left(e^{\frac{4}{25}t} + 1\right) = 4\left(e^{\frac{4}{25}t} - 1\right)$ $v = \frac{4(e^{\frac{4}{25}t} - 1)}{(e^{\frac{4}{25}t} + 1)}$
We consider how the fraction will behave as t changes in order to find its range of values. With this, we can find the limiting value of the velocity. 	Consider $\frac{(e^{\frac{4}{25}t} - 1)}{(e^{\frac{4}{25}t} + 1)}$. For all real t, $(e^{\frac{4}{25}t} - 1) < (e^{\frac{4}{25}t} + 1)$, meaning the denominator always exceeds the numerator and hence $\left\|\frac{(e^{\frac{4}{25}t} - 1)}{(e^{\frac{4}{25}t} + 1)}\right\| < 1$ always. Therefore, $4\left\|\frac{(e^{\frac{4}{25}t} - 1)}{(e^{\frac{4}{25}t} + 1)}\right\| < 4 \rightarrow$ the cyclist won't exceed $4ms^{-1}$.
We use the relation $a = v \cdot \frac{dv}{dx}$ to find an expression for velocity in terms of displacement.	$a = \frac{16 - v^2}{50} = v \cdot \frac{dv}{dx} \rightarrow \frac{dv}{dx} = \frac{1}{50v}(16 - v^2) \rightarrow \frac{v}{16 - v^2}\mathrm{d}v = \frac{1}{50}\mathrm{d}x$ The numerator is a function of the differential of the denominator so $\int \frac{v}{16 - v^2}\mathrm{d}v = \int \frac{1}{50}\mathrm{d}x \rightarrow -\frac{1}{2}\ln\|16 - v^2\| = \frac{1}{50}x + C$
We use the initial conditions to find this new integration constant. As the cyclist starts from rest at the origin and the function is increasing for v>0, we can simply sub in v=2.	When x$= 0, v = 0 \therefore \ln\|16\| = -2C \rightarrow C = -\ln(4) \rightarrow \frac{1}{50}x = \frac{1}{2}\ln(16) - \frac{1}{2}\ln\|16 - v^2\|$ $\rightarrow x = 25\ln\left\|\frac{16}{16 - v^2}\right\|$ When $v = 2, x = 25\ln\left\|\frac{16}{16 - 2^2}\right\| = 25\ln\left\|\frac{16}{12}\right\| = 25\ln\left\|\frac{4}{3}\right\|$ The cyclist travels $25\ln\left\|\frac{4}{3}\right\|$ m.