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Further Centres of Mass Cheat Sheet

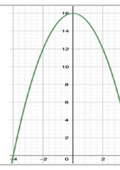
Edexcel Further Mechanics 2

In this chapter, you will learn how to find the position of the centre of mass for 2d/3d shapes by integrating a mass distribution function. Note: For this chapter, integration is not being assessed so it can be done on a calculator.

Centre of mass of a 2D shape

- We can use the equation of a graph/function to find it's centre of mass. For 2D shapes, these are called laminas of negligible thickness.
- Integration is the sum of an infinite number of strips with height y and length δx. Using this and the 1D center of mass formula for an n-particle system, $\sum m_i x_i = \bar{x} \sum m_i \; where \; i = 1, 2, \dots n$, we get the following results which you should learn (as they can be quoted without proof in exams):
- For a 2D model with equation f(x): $\bar{x} = \frac{\int_a^b x(f(x))\,dx}{\int_a^b f(x)\,dx}$ $\quad \bar{y} = \frac{\int_a^b \frac{1}{2}((f(x))^2\,dx}{\int_a^b f(x)\,dx}$
  For a 2D model with equation f(y): $\bar{y} = \frac{\int_a^b y(f(y))\,dy}{\int_a^b f(y)dy}$ $\quad \bar{x} = \frac{\int_a^b \frac{1}{2}((f(y))^2\,dy}{\int_a^b f(y)\,dy}$
- We integrate between the required limits a and b to give us the coordinates of the centre of mass. The values a and b will be given in the question or can be worked out using the equation of the graph.

Example 1: Find the coordinates of the centre of mass of a uniform lamina formed from the graph $y = 16 - x^2$ between the positive x and y axes.

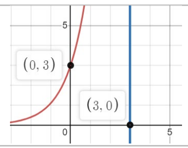
Start by sketching graph if it is not given in the question as this will help to find the limits for the integration, a and b.	
Reading the question, since we are interested in the graph between the positive x and y axis, we can see that we need to integrate between 0 and 4 to give us the $\int_a^b f(x)dx$ part of the formula. This gives us the area.	$\int_0^4 (16 - x^2)\,dx = \frac{128}{3}$
Now using the formula $\bar{x} = \frac{\int_a^b x(f(x))\,dx}{\int_a^b f(x)\,dx}$ from above we can calculate $\bar{x}$.	$\int_a^b xf(x)\,dx = \int_0^4 x(16 - x^2)\,dx = 64$ $\bar{x} = \frac{\int_a^b x(f(x))\,dx}{\int_a^b f(x)\,dx} = \frac{64}{128 \div 3} = 1.5$
Now using the formula $\bar{y} = \frac{\int_a^b \frac{1}{2}((f(x))^2\,dx}{\int_a^b f(x)\,dx}$ from above we can calculate $\bar{y}$.	$\int_0^4 (16 - x^2)^2\,dx = \frac{8192}{15}$ $\bar{y} = \frac{\int_a^b \frac{1}{2}((f(x))^2}{\int_a^b f(x)} = \frac{8192 \div 15}{128 \div 3} = 12.8$
For completeness, we state our final answer.	The centre of mass is $(\bar{x}, \bar{y}) = (1.5, 12.8)$

Centre of mass of 3D shapes

We can use volumes of revolutions to find the centre of mass of a 3D shape starting from a graph. This will create different shapes which depend on the shape of the graph.

- Using the same logic as for the 2D shapes: instead of 2D stripes we are now using 3D discs which are formed as a result of rotating a strip about an axis (see the topic of Volume of Revolutions from Core Pure 1).
- The coordinate of the centre of mass of the axis which you do not rotated around will be 0 due to it being a line of symmetry. This is as the centre of each disc would be at $x = 0$ if you rotate around the y-axis and at $x = 0$ if you rotate around the y axis.
- Again using $\sum m_i x_i = \bar{x} \sum m_i \; where \; i = 1, 2, \dots n$, we get the following results which you should learn (these can also be quoted without proof in exams):
- For a 3D model with equation $f(x)$:
- For rotation around the x-axis:
  - $\bar{x} \int_a^b (f(x))^2\,dx = \int_a^b x(f(x))^2\,dx$
  - $\bar{y} = 0$ as is the axis of symmetry
- For rotation around the y-axis:
  - $\bar{y} \int_a^b (f(y))^2\,dy = \int_a^b y(f(y))^2\,dy$
  - $\bar{x} = 0$ as is the axis of symmetry
- Note π has been excluded from the above equations as it cancels out

Example 2: Find the centre of mass of the solid revolution formed when the region for the curve with equation $y = 3e^x$ is rotated around the x axis between $x = 0$ and $x = 3$

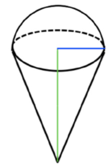
Sketch the graph as this will help know the shape formed when the graph is rotated.	
Rearrange the formula $\bar{x} \int_a^b (f(x))^2\,dx = \int_a^b x(f(x))^2\,dx$ above to calculate $\bar{x}$.	$\bar{x} = \frac{\int_a^b x(f(x))^2 dx}{\int_a^b (f(x))^2\,dx} = \frac{\int_0^3 x(3e^x)^2\,dx}{\int_0^3 ((3e^x)^2\,dx} = 2.51$ (to 3 d.p.)
State the full coordinates of the centre of mass ($\bar{y} = 0$)	$(\bar{x}, \bar{y}) = (2.51, 0)$

Centre of mass of composite 3D bodies

- These questions will normally involve two different shapes on top of each other whose centre of mass can be worked out using volumes of revolution or from the formula booklet.
- Once we have worked out the centre of mass of each object, we choose an origin (this should either be at the connecting points of the objects or at the base) before using $\sum m_i x_i = \bar{x} \sum m_i$ to find the overall centre of mass, taking each body as a point mass concentrated at its Centre of Mass.
- We assume uniform density for the shapes (unless otherwise stated) meaning we just use the volume. (calculated using $\pi \int_a^b (f(x))^2\,dx$) as our mass (i.e. mass is proportional to volume). This is normally shown in a question by the comment 'made out of the same material.'
- This approach also applies to bodies formed by subtracting one standard shape from another: subtracted shapes have 'negative' mass.
- To keep track of the different COMs and masses, a table can be very helpful as seen in the example below

Example 3: Find the centre of mass of the following body: A solid cone of radius 8, height 4, mass 3kg, placed on the plane face of a hemisphere mass 2kg, radius 8.

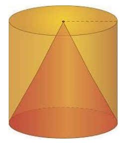
Object	Cone	Hemisphere
Mass	3	2
CoM (using the join of the object as y= 0)	From the formula booklet: $\bar{y} = \frac{1}{4}h = \frac{1}{4}(4) = 1$	From the formula booklet: $\bar{y} = \frac{3}{8}r = \frac{3}{8}(8) = 3$ But we take $\bar{y} = -3$ as it is below the plane face where our origin lies.



Now use $\sum m_i x_i = \bar{x} \sum m_i$ but in this case $\sum m_i y_i = \bar{y} \sum m_i$	$(3 + 2)\bar{y} = 3 \times 1 + 2 \times -3$ $\bar{y} = -0.6$
State full coordinates	$\bar{x} = 0$ because both objects are symmetrical around the y axis so $(\bar{x}, \bar{y}) = (0, -0.6)$

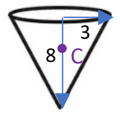
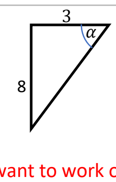
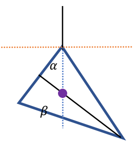
Example 4: Find the centre of mass of the following composite shape: A solid cylinder standing upright with radius 5 cm and height 8 cm where a solid cone with radius 5 cm and height 8 cm made out of the same material has been removed

Object	Cone	Cylinder
Mass	$V = \frac{\pi r^2 h}{3} = -\frac{200}{3}\pi$ (negative mass)	$V = \pi r^2 h = 200\pi$
COM (using the base of the cylinder as y= 0)	From the formula book: $\bar{y} = \frac{1}{4}h = 2$	From the formula book: $\bar{y} = \frac{h}{2} = 4$



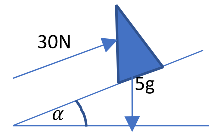
Now Use $\sum m_i y_i = \bar{y} \sum m_i$	$\left(200\pi - \frac{200\pi}{3}\right)\bar{y} = -\frac{200\pi}{3} \times 2 + 4 \times 200\pi$ $\bar{y} = 5$
State full coordinates	$\bar{x} = 0$ because both objects are symmetrical around the y axis so $(\bar{x}, \bar{y}) = (0,5)$

Example 5: A uniform solid cone has radius 3 cm and height 8 cm. When is it suspended from a point P which is on the edge of it's circular base, what angle does its base make with the vertical when in equilibrium?

Draw and label a diagram (let C be the COM)	 Since the COM is now directly below P, OC is our vertical. This means a is now our angle between the shape and the vertical.
When suspended, the CoM of the cone is vertically below the point it is suspended from. To find the angle the base makes with the horizontal (β) we need to work out the angle α, which depends on the location of the Centre of Mass.	  We want to work out the angle α and since we have the opposite and adjacent side, we know $\tan\alpha = \frac{2}{3} = 33.7°$.
Write the final answer in degrees unless otherwise stated.	β = 90 – 33.7° = 56.3° (to 1 d.p.)

- For when put on an inclined plane at angle α, by resolving forces parallel and perpendicular to the plane, we are able to know the force required in order to prevent the object from going down the plane
- Note if this parallel component of the force is larger than parallel component of the object, then it will go up the plane instead. However, if they are equal they will remain in equilibrium.

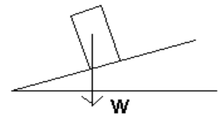
Example 6: A force of magnitude 30N is acting parallel to a smooth slope inclined at angle α where $\tan\alpha = \frac{3}{4}$. A cone, with mass 5kg, radius 2cm, and height 16cm has its base face on the plane. Determine whether the cone moves up or down the plane

Draw and label the diagram.	
Work out the different components of the weight	Using $\tan\alpha = \frac{3}{4}$: $\sin\alpha = \frac{3}{5}$ $\cos\alpha = \frac{4}{5}$ The parallel component of the weight is $5g\sin\alpha = 3g$.
Now resolve forces	$30N - 3g = 0.6N$
State final answer and its explanation	The cone will move up the plane as there is a resultant force of $0.6N$ up the plane.

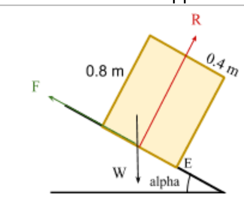
Points of Toppling and Sliding

We can work out where these points are by resolving forces or taking moments about a point. A good diagram here is helpful as it makes it easier to see where the different forces are acting.

- **The point of sliding** is where the frictional force
  $(\mu \times \text{perpendicular component of weight})$
  from the plane is equal to the parallel component of the weight (assuming no other external forces are acting on the object). This means when we place an object of mass m onto a rough plane with coefficient of friction μ inclined at angle α, the point of sliding is $F = \mu R$.
- **The Point of toppling** is where the COM of an object is vertically above one of the edges of the object. We can work out the angle at which this happens as looking at the diagram, if we know the COM and the radius of this object, we can work out the angle α and hence know the angle at which it will topple.



Example 7: A wooden block mass 4kg, with height 0.8m and radius 0.4m is put on a slope with coefficient of friction 0.4. The angle at which the slope is inclined, α, is slowly increased until either the block topples or slides. Determine whether the block will topple or slide first.

Draw a diagram to help identify the forces which are acting.	
Using $F = \mu R$ work out the point of sliding (assume that the block is in equilibrium)	$Weight\ down\ the\ slope = 4g\sin\alpha$ = Friction = $\mu 4g\cos\alpha$ $\tan\alpha = \mu$ $\alpha = 21.8°$ (the angle at which the block will slide)
Work out the point of toppling using the diagram to work out what the angle α would need to be	$\tan\alpha = \frac{0.4}{0.8}$ $\alpha = 26.6°$ This is the angle at which the block will topple.
Evaluate the final answer	$21.8° > 26.6°$, therefore the block will slide first before it topples.

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