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Edexcel FM2

Centres of Mass of Plane Figures Cheat Sheet

In this chapter, you will learn how to find the position of the centres of mass of different shapes and systems, including composite systems. You will learn about the different physical forms we use in mechanics and the importance of defining axes and origins.

Centre of mass of a set of particles on a straight line

We can use **moments** to find where a centre of mass lies on a straight line of particles. Recall that a moment is given by $mass \times distance\ from\ pivot$, and the centre of mass balances out with the sum of the moments along the straight line:

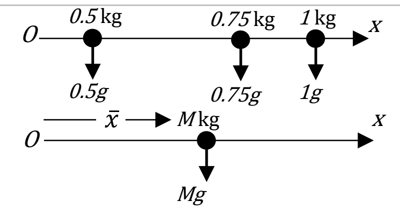
Centre of mass position

- $\sum m_i x_i = \bar{x} \sum m_i$ *where $i = 1, 2, \ldots, n$ for an n-particle system.*

A rod is a 1D shape with negligible thickness and uniformly distributed weight. A particle is a 1D shape with negligible size and negligible weight.

When solving problems like these, choose your origin, your axis, and pivot point. You need to hold these constant throughout the problem.

Example 1: Three ornaments, with masses 500 g, 750 g, and 1 kg, are displayed along a mantelpiece. They are positioned at (1,0), (5,0), and (7,0) respectively. Model the mantelpiece as a massless rod and the ornaments as particles. Find the centre of mass of this display.

Start by sketching out the problem. Draw a diagram showing each particle and its weight, then a second with the total weight Mg acting at the centre of mass (recall that the centre of mass is the point at which the total weight of an object effectively acts).	
We find M by equating the vertical forces in each diagram. Remember to keep your working in the same units and note that the g cancels.	$0.5g + 0.75g + 1g = Mg$ $\therefore M = 2.25$
We choose our pivot to be the origin, O, at the left-hand side of the mantelpiece. This is the choice that makes our job easiest, as the question gives us the positions in reference to this same origin. We hence take moments about O and use the equation above to find the centre of mass, substituting in with our value for M:	$(0.5g \times 1) + (0.75g \times 5) + (1g \times 7) = Mg \times \bar{x}$ $(0.5 \times 1) + (0.75 \times 5) + (1 \times 7) = 2.25\,\bar{x}$ $0.5 + 3.75 + 7 = 2.25\,\bar{x}$ $5 = \bar{x}$
For completeness, we state our final answer.	The centre of mass is (5,0).

We can check whether our answer is reasonable by considering the layout of mass on the mantelpiece. Most of the mass is on the right-hand side, beyond the halfway mark (3.5,0), so we expect our centre of mass to be on the right, too.

Centre of mass of a set of particles arranged in a plane

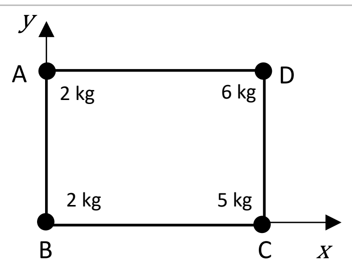
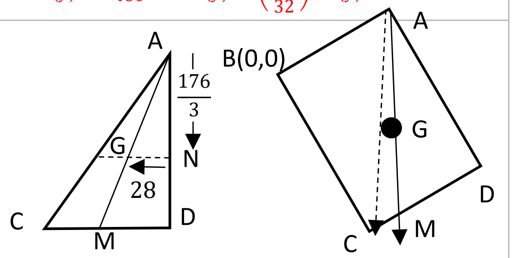
Like with force diagrams, we find the centre of mass of a plane of particles by resolving the positions of each into perpendicular components. We then consider the two coordinates/axes separately, using the fact that the sum of the moments is balanced by the centre of mass in each direction:

Position vector of centre of mass

- $\sum_i^n m_i x_i = \bar{x} \sum_i^n m_i$ *and* $\sum_i^n m_i y_i = \bar{y} \sum_i^n m_i$ for a system of n-particles
- For a general coordinate system, if each particle has its own position vector $\boldsymbol{r}$, then $\sum_i^n m_i \boldsymbol{r}_i = \bar{\boldsymbol{r}} \sum_i^n m_i$

If the question doesn't provide specific axes or coordinates, you should choose your own, as before. Choose two directions that are perpendicular to each other and that will minimise your workload – don't overcomplicate things!

Example 2: A rectangular plate ABCD is hung from a peg at A, and particles of masses 2 kg, 2 kg, 5 kg and 6 kg attached at points A, B, C and D respectively. The side AB has length 60 cm and side BC has length 80 cm. Find the angle the line AC makes with the vertical when the plate is let go and comes to rest. Assume the mass of the plate is negligible.

Sketch the problem out and work out what the question is asking for, how you will get there, and which choice of axes will simplify the problem most.	 Let A be at (0,60), B(0,0), C(80,0), and D(80,60) When an object hangs freely, its weight acts through its centre of mass, such that its centre of mass is directly below the point of suspension. This means when we find the position of the centre of mass, we can find the angle the plate hangs at and hence the angle AC makes with the vertical.
We use the above equality to find the position of the centre of mass, writing in vector form.	$2\binom{0}{60} + 2\binom{0}{0} + 5\binom{80}{0} + 6\binom{80}{60} = (2+2+5+6)\binom{\bar{x}}{\bar{y}} \to \binom{880}{480} = 15\binom{\bar{x}}{\bar{y}} \to \begin{pmatrix}\frac{176}{3}\\32\end{pmatrix} = \binom{\bar{x}}{\bar{y}}$
Now we have the centre of mass, we can draw out how the plate will hang freely. We use trigonometry to calculate the angle the question asks for. We can use the same coordinates as before, because what we're doing is equivalent to rotating the whole coordinate system.	The angle we want is CAM. The vector AG is $\binom{0}{60} - \begin{pmatrix}\frac{176}{3}\\32\end{pmatrix} = \begin{pmatrix}-\frac{176}{3}\\28\end{pmatrix}$ We use the similar triangle AGM to find angle MAD: $\theta_{MAD} = \arctan\left(28 \div \frac{176}{3}\right) = \arctan\left(\frac{21}{44}\right)$ We use triangle CAD to find angle CAD: $\theta_{CAD} = \arctan(60 \div 80) = \arctan\left(\frac{6}{8}\right)$ $\theta_{CAM} = \theta_{CAD} - \theta_{MAD} = 11.36°$ 
We write our answer clearly.	The line AC makes an angle 11.36° with the vertical.

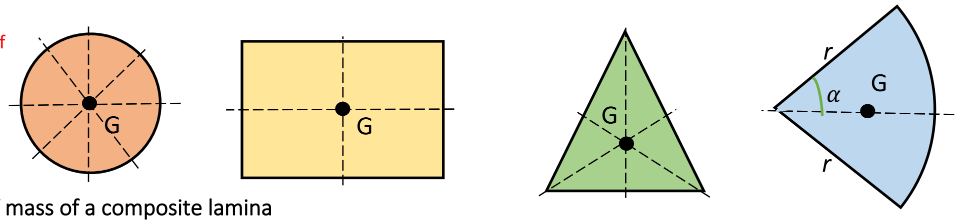
Centres of mass of standard uniform plane laminas

A **lamina** is an object viewed as having area but no volume. Laminas are used to model objects where one dimension (thickness) is very small compared with the other two (length and width), e.g. a piece of card. If its mass is distributed evenly through its area, we call the lamina **uniform**. The uniform lamina's centre of mass always lies on its axis of symmetry - meaning if it has more than one, it is located at the intersection of these axes. We can apply this knowledge to a variety of shapes:

- **Uniform circular disc:** Every diameter is a line of symmetry. Their intersection is the centre of the disc.
- **Uniform rectangular lamina:** The two lines of symmetry each join the midpoints of a pair of opposite sides. The centre of mass lies where these bisectors cross.
- **Uniform triangular lamina:** A triangle only has lines of symmetry if it is isosceles or equilateral. However, it is true for all triangles that the centre of mass lies at the centroid of the triangle. The centroid is where the medians, the lines joining a vertex to the midpoint of the opposite side, meet.
    - Say the points of a uniform triangular lamina are $(x_1, y_1), (x_2, y_2), \& (x_3, y_3)$. In this case, the centre of mass' coordinates are found by taking the mean of these coordinates, i.e. G (the centre of mass) is at $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.
- **Uniform sector of a circle:** In radians, for a sector of radius r and centre angle 2α, we find the centre of mass is a distance $\frac{2r\sin\alpha}{3\alpha}$ along the axis of symmetry

This is explored further in Section 3.1

The centres of mass for different uniform laminas



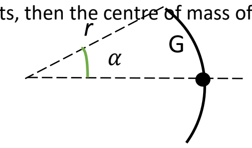
Find the C.O.M. for each component, then use $\sum_i^n m_i \boldsymbol{r}_i = \bar{\boldsymbol{r}} \sum_i^n m_i$

Centre of mass of a composite lamina

You may be asked to find the centre of mass of a **composite uniform lamina**. A composite uniform lamina is formed when two standard uniform laminas are joined together. In this case, find the centre of mass by considering each part as a particle acting at its centre of mass, with its mass being proportional to its area (as mass is distributed evenly, the whole lamina has the same mass per unit area). Split the lamina into standard shapes, find the positions of their centres of mass, then replace the shapes with particles of respective mass located at the positions just calculated. This approach also applies to laminas formed by subtracting one standard shape from another: subtracted shapes have 'negative' mass.

Centre of mass of a framework

A **framework** is a system made of a number of rods or pieces of wire joined together. Similar to a composite lamina, you can find the centre of mass of a framework by considering the centres of mass of each individual rod or wire. If you have a particle with corresponding mass acting at each of these points, then the centre of mass of the entire framework is equivalent to the centre of mass of this system of particles. There are a few more shapes you may encounter:

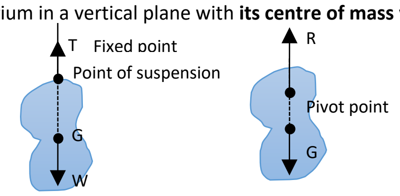


- **Uniform straight rod:** Recall the centre of mass of a uniform straight rod is at its midpoint
- **Uniform circular arc:** In radians, for an arc of radius r and centre angle 2α, the centre of mass is located on its axis of symmetry, a distance $\frac{r\sin\alpha}{\alpha}$ from the centre.

Lamina and frameworks in equilibrium

We can use our understanding of centres of mass to analyse how objects behave when suspended from a point, for example attached to a piece of string or allowed to rotate about a pivot. In these cases, there only two forces acting on the object: its weight and the tension or reaction force, depending on the type of suspension. When we suspend a lamina or framework freely from a point or balance it on a pivot, it will come to equilibrium in a vertical plane with **its centre of mass vertically below the fixed point**.

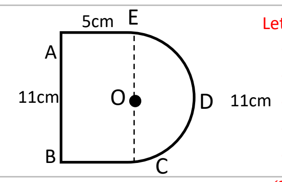
You can save time by drawing a line between the suspension point and the C.O.M. and marking it as the vertical, instead of redrawing the shape.



Non-uniform composite laminas and frameworks

You can also be asked about non-uniform laminas and frameworks, where the mass is distributed differently across the area. The same methods used in this chapter apply here. Always approach a problem by splitting the shape into component parts that you know how to deal with and replacing each segment with a particle of corresponding mass acting through the centre of mass of the segment.

Example 3: A framework is made from two pieces of uniform wire bent into the shape shown below. The wire EABC forms three sides of a rectangle and has a mass per unit length twice that of the wire EDC, which is a semicircle centred at O. Find the centre of mass.

Using the diagram provided, we define our origin and axes.	 Let the origin be O and our axes be along OD and BA: • A $(-5,11)$ • B $(-5,-11)$ • C $(0,-11)$ • D $(11,0)$ • E $(0,11)$
Use the formula above to find the centre of mass of the semicircle. As the shape is symmetrical about the x-axis (OD), we deduce that its centre of mass has the coordinate $y = 0$.	$x = \frac{r\sin\alpha}{\alpha} \therefore x = \frac{11 \times \sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \frac{22}{\pi} \Rightarrow G_{semicircle} = \left(\frac{22}{\pi}, 0\right)$
The centres of mass of the straight wires are at their midpoints.	$G_{AE}\ (-2.5, 11), G_{AB}\ (-5, 0), \&\ G_{BC}\ (-2.5, -11)$
We replace each G with a particle acting at that point, using that the mass per unit length of the straight wires is double that of the arc. We then use the formula for a coplanar set of particles to find the centre of mass of the total framework.	Length of EDC $= 11\pi$ $11\pi\begin{pmatrix}\frac{22}{\pi}\\0\end{pmatrix} + 10\binom{-2.5}{11} + 44\binom{-5}{0} + 10\binom{-2.5}{-11} = (64 + 11\pi)\binom{\bar{x}}{\bar{y}}$ $\binom{242 - 25 - 220 - 25}{110 - 110} = \binom{-28}{0} = (64 + 11\pi)\binom{\bar{x}}{\bar{y}}$ $\binom{\bar{x}}{\bar{y}} = \begin{pmatrix}-\frac{28}{64 + 11\pi}\\0\end{pmatrix}$

Mass is proportional to length, so the value of mass/length will cancel through. The important info to include in our expression is the proportionality constant: the length in the case of the semicircle and double the length for the straight wires.