Circular Motion Cheat Sheet

Edexcel FM2

Previously, you will have come across many problems involving linear motion, or motion in a single direction. We will now learn to solve problems involving circular motion. The key is that for a body to move in circular motion the resultant force acting on the body should be toward the centre of the circle and that energy is conserved throughout the duration of the motion. Problems in this chapter may involve strings, wires, circular paths, etc. with a mass attached to them. Some problems may involve a further part wherein the mass can lose connection with the circle. In these cases, you need concepts of projectile motion to solve the problem.

Angular Speed

Angular Speed is a measure of the rate at which the angle of a rotating body changes. This angle is measured in radians. The SI unit of the angular speed is rad s^{-1}. Angular speed can be shown using the symbol $\dot{\theta}$, which represents the derivative of the angle with respect to time. ω is more commonly used to represent angular speed of a body.

$$\omega = 2\pi f = \frac{2\pi}{T} \ (as\ f = \frac{1}{T})$$

Other units for angular velocity include revolutions per minute (rpm). To use in formula, we need to convert this into rad s^{-1}. We can do this multiplying rpm with $\frac{2\pi}{60}$.

The angular velocity can also be given in terms of the linear velocity, v of an object and the radius, r of the circular path it is moving in.

$$\omega = \frac{v}{r}$$

Horizontal circular motion

When moving in a horizontal circular path, the objects velocity is changing continuously because it's changing direction. Since the velocity is changing, the object must be accelerating.

$$a = \omega^2 r = \frac{v^2}{r}$$

This acceleration is always toward the centre of the circular path that the object is traveling in. This acceleration could be a result of:

- Tension in string keeping the object in a circle.
- Friction between surface and object.
- Sum of component of friction and component of reaction in case of a banked track.

By Newton's second law *F = ma*. This implies that in case of a horizontal circle, the force towards the centre of the circle is:

$$F = ma = \frac{mv^2}{r} = m\omega^2 r$$

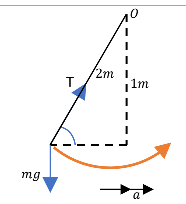
This force is called the centripetal force and it is a resultant force responsible for keeping the object revolving in circular motion.

Horizontal circle problems:

The most important part with any of these questions is to make a good diagram. After this we need to resolve force towards the centre of the horizontal circle and then set them equal to the centripetal force.

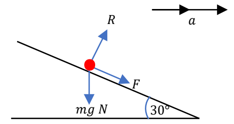
- Conical Pendulum
  - The key with these types of the questions is to find the force of tension in the string by using Newton's second law and then setting the horizontal component of the tension equal to the centripetal force.

Example: A bob of mass 4 kg is attached to a light inextensible string of length 2 m to a fixed point O. The bob is made to move in a horizontal circle whose centre lies 1 m vertically below O. Find the tension in the string and the speed of the bob.

We must start off by making a diagram first.	
Find radius of the circular path the bob moves in.	Using Pythagoras: $r^2 = 2^2 - 1^2$ ⇨ $r = \sqrt{3}$
Apply NEWTON'S SECOND LAW vertically.	$\cos\theta = \frac{\sqrt{3}}{2}; \sin\theta = \frac{1}{2}$ $mg = T\sin\theta$ ⇨ $2mg = T$ ⇨ $T = 8g = 78.4\ N$
Apply NEWTON'S SECOND LAW horizontally.	$T\cos\theta = \frac{mv^2}{r}$ ⇨ $8g\ \frac{\sqrt{3}}{2} = \frac{4v^2}{\sqrt{3}}$ ⇨ $3g = v^2$ ⇨ $v = 5.42\ \text{ms}^{-1}\ (3sf)$

- **Banked Track**
  - To solve questions involving banked tracks, we must take the resultant of the component of force of friction and component of reaction force towards the centre of the circle.
  - This resultant must then be set equal to centripetal force.
  - What is crucial in these questions is to realize that if the body is too fast on the track, the friction acts down the slope and if it is too slow, the friction acts up the slope.

Example: A truck moves around a bend of radius 100 m, banked at an angle of 30° to the horizontal. The truck is moving at a speed of 20 ms^{-1}. What is the least possible value of the coefficient of friction if the truck does not slip up the slope?

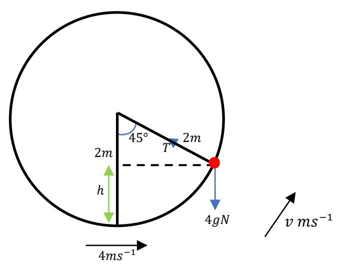
We must start off by making a diagram.	
We should now apply NEWTON'S SECOND LAW vertically.	$R\cos(30) = mg + F\sin(30)$ $F = \mu R$ ⇨ $R\cos(30) - \mu R\sin(30) = mg$ ⇨ $R(\cos(30) - \mu\sin(30)) = mg$
Apply NEWTON'S SECOND LAW horizontally.	$R\sin(30) + F\cos(30) = \frac{mv^2}{r}$ ⇨ $R\sin(30) + \mu R\cos(30) = \frac{400m}{100} = 4m$ ⇨ $R(\sin(30) + \mu\cos(30)) = 4m$
Both equations have an unknown μ. Therefore, solve for μ by dividing both equations.	$\frac{\cos(30) - \mu\sin(30)}{\sin(30) + \mu\cos(30)} = \frac{g}{4}$ ⇨ $9\cos(30) - 9\mu\sin(30) = g\sin(30) + g\mu\cos(30)$ ⇨ $9\cos(30) - g\sin(30) = \mu(9\sin(30) + g\cos(30))$ ⇨ $\mu = \frac{9\cos(30) - g\sin(30)}{9\sin(30) + g\cos(30)} = 0.223\ (3dp)$

Vertical Circular Motion

The key with these questions is, again, to make a good diagram. After this we typically apply conservation of energy between two points on the circle and then we use Newton's second law toward the centre of the circle. Conservation of energy is used because the mass gains/loses height during the motion and hence its energy stores change. Depending on whether the mass leaves the circle, we may need to further use principles of projectile motion to solve questions.

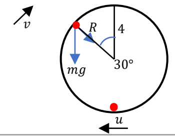
- **Mass on String/Rod**
  - An important thing to remember is that the mass will perform circular motion only if its velocity at the top of the circle is greater than 0 i.e. $v_{top} > 0$.
  - First apply conservation of energy to obtain an expression for v^2.
  - Apply Newton's second law toward the centre of the circle and substitute the above obtained expression in this equation.

Example: A particle P of mass 4 kg is attached to the end of a light inextensible string, the other end of which is fixed at a point O. The length of the string is 1 m. When the particle is hanging directly below O, it is projected horizontally with speed 4 ms^{-1}. Find the speed of the particle and the tension in the string when the particle has moved through an angle of 45 degrees.

First step is to draw a diagram.	
Apply Conservation of Energy.	$Initial\ KE + Initial\ PE = Final\ KE + Final\ PE$ ⇨ $\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mgh$ ⇨ $\frac{1}{2}(4)^2 + 0 = \frac{1}{2}v^2 + g(1 - 1\cos(45))$ ⇨ $8 = \frac{1}{2}v^2 + g - \frac{g\sqrt{2}}{2}$ ⇨ $16 = v^2 + 2g - g\sqrt{2}$ ⇨ $v^2 = 16 - 2g + g\sqrt{2} = 3.20\ \text{ms}^{-1}$
Apply NEWTON'S SECOND LAW towards centre of circle.	⇨ $T - 4g\cos(45) = \frac{mv^2}{r}$ ⇨ $T - \frac{4g\sqrt{2}}{2} = 4(16 - 2g + g\sqrt{2})$ ⇨ $T = 68.8\ \text{N}$

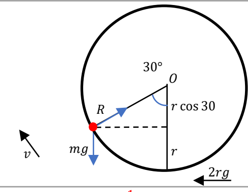
- **Mass moving on the inside surface of a track**
  - The main force acting is the reaction force between the surface and the mass. This force acts towards the centre of the circle.
  - The sum total of the reaction force and the component of the mass acting towards the centre of the circle is equal to the centripetal force.
- **Mass moving on the outside surface of a track**
  - In this case the normal reaction force acts outwards. At some point the weight of the particle may be insufficient to keep the particle moving in a circle in which case it might fall.

Example: A particle P is free to move on the smooth inner surface of a fixed thin hollow sphere of internal radius 4m and centre O. The particle passes through the lowest point of the spherical surface with speed Ums^{-1}. The particle loses contact with the surface when OP is inclined at an angle of 30° to the upward vertical. Find U.

We must start off by making a diagram.	
Apply conservation of energy.	⇨ $\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mgh$ ⇨ $\frac{1}{2}U^2 + 0 = \frac{1}{2}v^2 + g(4 + 4\cos 30)$ ⇨ $U^2 = v^2 + 2g(4 + 4\cos 30)$ ⇨ $v^2 = U^2 - 2g(4 + 4\cos 30)$
Apply NEWTON'S SECOND LAW towards the centre at the point where mass leaves the surface.	$mg\cos 30 - R = \frac{mv^2}{r}$ At the point where contact is lost, $R = 0$: $mg\cos 30 = \frac{m(U^2 - 2g(4 + 4\cos 30))}{4}$ ⇨ $g\cos 30 = \frac{U^2 - 8g - 8g\cos 30}{4}$ ⇨ $12g\cos 30 + 8g = U^2$ ⇨ $U = 13.4\ \text{ms}^{-1}$

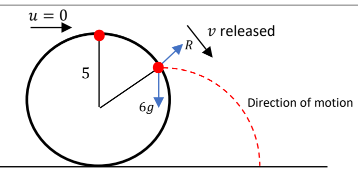
- **Mass/ Bead threaded onto a wire**
  - In this case again, the velocity at top must be greater than 0 for the bead to complete circular motion.
  - If the velocity of the particle is not high enough the particle will oscillate in the ring.

Example: A small bead A of mass 4kg is threaded on a fixed smooth circular wire in the vertical plane. The wire has centre O and radius r. The bead is projected from the lowest point of the wire with speed $(2\sqrt{rg})$. Find the speed of the bead and the reaction of the wire on the bead when the bead makes an angle of 30° with the downward vertical

First step is to draw a diagram	
Apply Conservation of Energy.	⇨ $\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mgh$ ⇨ $\frac{1}{2}(4rg) + 0 = \frac{1}{2}v^2 + g(r - r\cos 30)$ ⇨ $2rg = \frac{1}{2}v^2 + gr - \frac{gr\sqrt{3}}{2}$ ⇨ $4rg = v^2 + 2gr - gr\sqrt{3}$ ⇨ $v^2 = 4rg - 2gr + gr\sqrt{3} = gr(2 - \sqrt{3})\ \text{ms}^{-1}$
Apply NEWTON'S SECOND LAW towards centre of circle.	⇨ $R - 4g\cos 30 = \frac{4(4rg - 2gr + gr\sqrt{3})}{r}$ ⇨ $T - \frac{4g\sqrt{3}}{2} = 4(8g - 2g + g\sqrt{3})$ ⇨ $T = 337\ N$

If the object/mass is not held in its circular path by any contact force, such as tension, normal reaction, or friction, then as soon as the contact force breaks, the particle is treated as a projectile. This is because at this stage, gravity is the only force acting on the particle.

Example: A smooth sphere with centre O and radius 5m is fixed to a horizontal surface. An object A of mass 6kg is at rest on the highest point of the sphere. The mass is moved slightly from here and it begins to slide down the sphere's surface. At what angle to the upward vertical does the mass leave the sphere? Find also, the magnitude of the velocity of mass when it hits the horizontal surface.

First step is to draw a diagram.	
Apply Conservation of Energy.	⇨ $\frac{1}{2}mu^2 + mgh = \frac{1}{2}mv^2 + mgh$ ⇨ $0 + g(10) = \frac{1}{2}v^2 + g(5 + 5\cos\theta)$ ⇨ $2(10g - 5g - 5g\cos\theta) = v^2$ ⇨ $v^2 = 10g - 10g\cos\theta$
Apply NEWTON'S SECOND LAW towards centre of circle.	⇨ $6g\cos\theta - R = \frac{mv^2}{r}$ ⇨ $6g\cos\theta = \left(\frac{6(10g - 10g\cos(\theta))}{5}\right)$ ⇨ $g\cos\theta = ((2g - 2g\cos\theta))$ ⇨ $3g\cos\theta = 2g$ ⇨ $\cos\theta = \frac{2}{3} => \theta = 48.2^o$
Find the velocity v at which mass leaves surface.	⇨ $v = \sqrt{10g - 10g\left(\frac{2}{3}\right)} = 5.72\ \text{ms}^{-1}$
Find height that the mass falls as a projectile.	⇨ $5 + 5\cos(\theta) = 25/3$
Resolve initial speed.	⇨ $vertical\ speed = v\sin\theta = v\frac{\sqrt{5}}{3} = 4.26$ ⇨ $horizontal\ speed = v\ cos\theta = 3.81$
Find final vertical velocity using SUVAT.	⇨ $v^2 = u^2 + 2ah$ ⇨ $v = \sqrt{4.26^2 + 2(-9.8)\left(-\frac{25}{3}\right)}$ ⇨ $v = 13.47$ and $horizontal\ v\ is\ constant = 3.81$
Find magnitude.	⇨ $\sqrt{13.47^2 + 3.81^2} = 14.0\ \text{ms}^{-1}$