Elastic Collisions in Two Dimensions Cheat Sheet

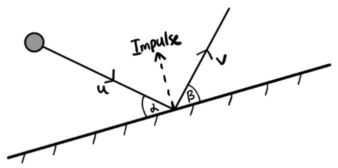
Edexcel FM1

In Chapter 4 of FM1, you learnt to solve problems involving the direct collision of particles. We will extend this further and learn to solve problems where the particles do not necessary move towards each other along a straight line. This is known as an oblique impact. Throughout this chapter, we use the assumption that any spheres and surfaces are smooth.

Oblique impacts with a fixed surface

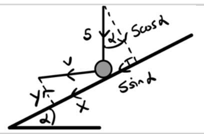
When a smooth particle strikes a smooth surface, the impulse acts perpendicular to the surface, through the centre of the sphere. As a result,

- The component of velocity parallel to the surface is unchanged.
  $\Rightarrow$ so $u\cos\alpha = v\cos\beta$
- You can use Newton's law of restitution to find the component of velocity of the sphere perpendicular to the surface.
  $\Rightarrow$ so $v\sin\beta = e(u\sin\alpha)$



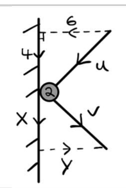
When solving problems involving impacts with a fixed surface, you should always consider motion parallel to the surface separately to the motion perpendicular to the surface.

Example 1: A small smooth ball is failing vertically. The ball strikes a smooth plane which is inclined at an angle α to the horizontal, where $\tan\alpha = \frac{1}{2}$. Immediately before striking the plane the ball has speed $5\ \mathrm{ms}^{-1}$. The coefficient of restitution between the ball and the plane is $\frac{1}{2}$.
Find the speed of the ball immediately after the impact.

We begin by drawing a diagram: We label the components of the final velocity perpendicular and parallel to the surface X and Y respectively.	
We know the component of velocity parallel to the surface is unchanged.	$X = 5\sin\alpha$
Apply Newton's law of restitution: $rebound\ speed = e \times approach\ speed$	$e(5\cos\alpha) = Y \Rightarrow \frac{5}{2}\cos\alpha = Y$
Since we know $\tan\alpha = \frac{1}{2}$, we can work out the values of $\sin\alpha$ and $\cos\alpha$.	$\tan\alpha = \frac{1}{2}$ $\Rightarrow \sin\alpha = \frac{1}{\sqrt{5}}, \quad \cos\alpha = \frac{2}{\sqrt{5}}$
Our values for X and Y become:	$X = 5\left(\frac{1}{\sqrt{5}}\right) = \sqrt{5}, \quad Y = \frac{5}{2}\left(\frac{2}{\sqrt{5}}\right) = \sqrt{5}$
Now we can work out the speed.	$speed = \sqrt{X^2+Y^2} = \sqrt{(\sqrt{5})^2 + (\sqrt{5})^2} = \sqrt{10}$

Some questions will require you to work with vectors. The same procedure applies but you need to be confident with the application of vectors.

Example 2: A small smooth ball of mass 2 kg is moving in the xy –plane and collides with a smooth fixed vertical wall which contains the $y - axis$. The velocity of the ball just before impact is $(-6\boldsymbol{i} - 4\boldsymbol{j})ms^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{3}$. Find:
(a) the velocity of the ball immediately after the impact
(b) the kinetic energy lost as a result of the impact.

(a) We begin by drawing a diagram. We label the components of the final velocity perpendicular and parallel to the surface X and Y respectively.	
We know the component of velocity parallel to the surface is unchanged.	$X = 4$
Apply Newton's law of restitution to the perpendicular velocity: $rebound\ speed = e \times approach\ speed$	$e(6) = Y \Rightarrow \frac{6}{3} = Y = 2$
Now we can find the speed.	$speed = \sqrt{X^2+Y^2} = \sqrt{(4)^2+(2)^2}$ $= 2\sqrt{5}$
(b) First find the initial speed. Then we calculate both the kinetic energy before and the kinetic energy after. The loss in K.E. is given by $K.E._{before} - K.E._{after}$.	$speed_{before} = \sqrt{(-6)^2+(-4)^2} = 2\sqrt{13}$ $K.E._{before} = \frac{1}{2}mv^2 = \frac{1}{2}(2)(2\sqrt{13})^2 = 52\ \mathrm{J}$ $K.E._{after} = \frac{1}{2}mv^2 = \frac{1}{2}(2)(2\sqrt{5})^2 = 20\ \mathrm{J}$
The loss in K.E. is given by $K.E._{before} - K.E._{after}$.	$\therefore loss\ in\ K.E. = 52 - 20 = 32\ \mathrm{J}$

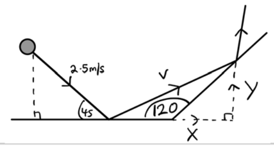
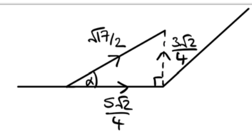
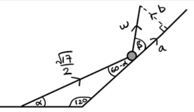
Successive oblique impacts

You also need to be able to solve problems involving successive oblique impacts of a sphere with a smooth surface.

- When solving questions involving successive impacts, you will need to consider each impact separately.

Example 3: Two smooth vertical walls stand on a smooth horizontal surface and intersect at an angle of $120°$. A smooth sphere of mass 0.1 kg is projected across the surface with speed $2.5\ \mathrm{ms}^{-1}$ at an angle of $45°$ to one of the walls and towards the intersection of the walls. The coefficient of restitution between the sphere and the walls is 0.6.

(a) Work out the speed and direction of motion of the sphere after the first collision.
The sphere then moves on to collide with the second wall.
(b) Calculate the kinetic energy of the sphere after the second collision.

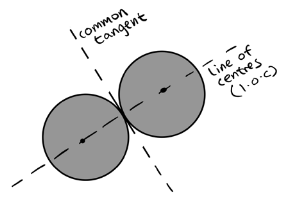
(a) We begin by drawing a diagram. We label the components of the final velocity (for the first collision) perpendicular and parallel to the surface X and Y respectively.	
We know the component of velocity parallel to the surface is unchanged.	$X = 2.5\cos 45 = \frac{5\sqrt{2}}{4}$
Apply Newton's law of restitution to the perpendicular velocity: $rebound\ speed = e \times approach\ speed$	$e(2.5\sin 45) = Y \Rightarrow (0.6)\left(\frac{5\sqrt{2}}{4}\right) = Y = \frac{3\sqrt{2}}{4}$
Now we can find the speed.	$speed = \sqrt{X^2+Y^2} = \sqrt{\left(\frac{5\sqrt{2}}{4}\right)^2 + \left(\frac{3\sqrt{2}}{4}\right)^2}$ $= \frac{\sqrt{17}}{2}\ \mathrm{ms}^{-1}$
To find the direction, we need to find the angle α. Constructing the velocity vector allows us to find α using basic trigonometry.	 $\Rightarrow \tan\alpha = \frac{\frac{3\sqrt{2}}{4}}{\frac{5\sqrt{2}}{4}} = \frac{3}{5}$ $\therefore \alpha = \tan^{-1}\left(\frac{3}{5}\right) = 30.96° = 31°\ (2\ s.f.)$
State the speed and direction.	So, the sphere moves with speed $\frac{\sqrt{17}}{2}\mathrm{ms}^{-1}$ at an angle of $31°$ to the surface.
(b) Draw a diagram for the second collision using the information we have found. We know the angle that the initial velocity makes is $60 - \alpha$, since angles in a triangle must add up to $180°$.	
We know the component of velocity parallel to the surface is unchanged.	$a = \frac{\sqrt{17}}{2}\cos(60-\alpha) = 1.80244\ldots$
Apply Newton's law of restitution to the perpendicular velocity: $rebound\ speed = e \times approach\ speed$	$b = 0.6\left(\frac{\sqrt{17}}{2}\sin(60-\alpha)\right) = 0.60036\ldots$
Now we can find the speed.	$speed = \sqrt{a^2+b^2}$ $= \sqrt{(1.80244\ldots)^2 + (0.60036\ldots)^2} = 1.8997\mathrm{ms}^{-1}$
Kinetic energy is given by $\frac{1}{2}mv^2$.	$K.E. = \frac{1}{2}(0.1)(1.8997\ldots)^2 = 0.180J$

As you can see, these questions do require a fair bit of working but are generally worth a lot of marks. The example above would typically be worth about 14 marks in total, for instance.

Oblique impacts between two smooth spheres

You also need to solve problems involving the oblique impact between two smooth spheres. You will need to use the following facts in such questions:

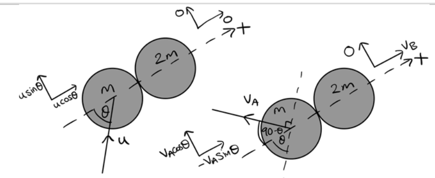
- The impulse acts along the line of centres of the two spheres.
- The principle of conservation of momentum can be applied parallel to the line of centres (l.o.c.).
- The components of velocities perpendicular to the line of centres are unchanged in the collision.
- Newtons law of restitution applies to the components of velocities parallel to the line of centres.



$C.L.M.$ also applies perpendicular to the l.o.c. but trivially so, since the components perpendicular are unchanged.

Any question involving an oblique impact between two smooth spheres will require you to use at least one of those four facts.

Example 4: A small smooth sphere A of mass m and a small smooth sphere B of the same radius but mass $2m$ collide. At the instant of impact, B is stationary and the velocity of A makes an angle θ with the line of centres. The direction of motion of A is turned through $90°$ by the impact. The coefficient of restitution between the spheres is e. Show that $\tan^2\theta = \frac{2e-1}{3}$.

We begin by drawing diagrams that detail the velocities before and after. We are told that the direction of motion of A is turned through $90°$. This allows us to find the angle that v_A makes with the line of centres in terms of θ.	
Apply $C.L.M.$ parallel to the line of centres. Divide through by m.	$m(u\cos\theta) + 2m(0) = m(-v_A\sin\theta) + 2m(v_B)$ $\Rightarrow u\cos\theta = 2v_B - v_A\sin\theta \quad [1]$
Apply Newton's law of restitution perpendicular to the line of centre: $rebound\ speed = e \times approach\ speed$	$e = \frac{v_B - -v_A\sin\theta}{u\cos\theta} = \frac{v_B + v_A\sin\theta}{u\cos\theta}$ $ue\cos\theta = v_B + v_A\sin\theta \quad [2]$
Next we want to use the fact that the velocity of A perpendicular to the line of centres is unchanged. This means $v_A\cos\theta = u\sin\theta$. We try to get an expression for v_A by eliminating v_B from [1] and [2].	$2\times[2]: \quad 2ue\cos\theta = 2v_B + 2v_A\sin\theta$ $-[1]: 2ue\cos\theta - u\cos\theta = 2v_B - 2v_B + 2v_A\sin\theta - -v_A\sin\theta$
Simplify and make v_A the subject.	$\Rightarrow u\cos\theta\,(2e-1) = 3v_A\sin\theta$ $\therefore v_A = \frac{u\cos\theta\,(2e-1)}{3\sin\theta}$
Now we use the fact that the velocity of A perpendicular to the line of centres is unchanged. This means $v_A\cos\theta = u\sin\theta$.	$v_A\cos\theta = u\sin\theta$ $\Rightarrow \frac{u\cos^2\theta\,(2e-1)}{3\sin\theta} = u\sin\theta$
Divide through by u.	$\Rightarrow \frac{\cos^2\theta(2e-1)}{3} = \sin^2\theta$
Divide through by $\cos^2\theta$ gives us the required result.	$\tan^2\theta = \frac{2e-1}{3}$

The above example gave us a scenario where two spheres collide obliquely and asked us to prove a relationship between θ and e. From a first glance, you may not immediately spot the correct approach for such a question since the result seems rather abstract and disconnected from the context given. It is important to remember that regardless of what you are asked to show or find, you will always need to use one or more of the four bullet points stated earlier. Using them will help you to get started and at the very least gain method marks, after which the next steps should become clear.

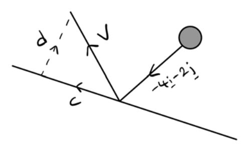
Problems where surfaces are given as equations

Some problems will require you to solve oblique impact problems where velocities are given as vectors, and the surface is given as a straight-line equation. These questions should still be approached in the same way as the earlier examples but finding the component of a velocity parallel/perpendicular to the surface requires slightly more work, and the use of the dot product which you met in the Vectors chapter (Ch. 9) from Core Pure 1. The following result is useful:

- The component of a vector $\boldsymbol{a}$ in the direction of another vector $\boldsymbol{b}$ is given by $\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}[\boldsymbol{b}]$.

Careful: $\boldsymbol{b}$ is a vector!

Example 5: A small smooth ball of mass 500 g is moving in the xy –plane and collides with a smooth fixed vertical wall which contains the line $x + y = 3$, The velocity of the ball just before the impact is $(-4\boldsymbol{i} - 2\boldsymbol{j})ms^{-1}$. The coefficient of restitution between the sphere and the wall is $\frac{1}{2}$. Find the velocity of the ball immediately after the impact.

A diagram is always helpful. We let $\boldsymbol{v} = \boldsymbol{c} + \boldsymbol{d}$, where $\boldsymbol{c}$ is the vector parallel to the wall and $\boldsymbol{d}$ the vector perpendicular to the wall.	
Find a vector in the direction of the line.	The wall contains the line $x + y = 3$. This line has gradient -1 so a vector in the direction of the line is $\begin{pmatrix}-1\\1\end{pmatrix}$.
Using the result bullet-pointed above, we can find c: $\boldsymbol{a} = velocity\ vector\ before$ $\boldsymbol{b} = vector\ in\ direction\ of\ wall$	So $\boldsymbol{c} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}[\boldsymbol{b}] = \frac{\begin{pmatrix}-4\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix}}{(-1)^2+(1)^2}\begin{pmatrix}-1\\1\end{pmatrix} = 1\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$
Next, we find a vector perpendicular to the line: This can be found by inspection; we just want a vector which when 'dotted' with $\begin{pmatrix}1\\1\end{pmatrix}$ gives 0.	A vector perpendicular to the wall is $\begin{pmatrix}1\\1\end{pmatrix}$. To confirm this, $\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\end{pmatrix} = -1 + 1 = 0$
Now to find $\boldsymbol{d}$ we need to use Newton's law of restitution.	The component of $\boldsymbol{v}$ perpendicular to the surface will be given by $e \times k$, where k is the component of $-4\boldsymbol{i} - 2\boldsymbol{j}$ perpendicular to the surface.
Using the result bullet-pointed above: $\boldsymbol{a} = velocity\ vector\ before,$ $\boldsymbol{b} = vector\ perpendicular\ to\ wall$	$\boldsymbol{d} = e\left[\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|^2}[\boldsymbol{b}]\right] = \frac{1}{2}\left[\frac{\begin{pmatrix}-4\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}{(1)^2+(1)^2}\begin{pmatrix}1\\1\end{pmatrix}\right] = \frac{-3}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-3/2\\-3/2\end{pmatrix}$
Add the vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ together to give $\boldsymbol{v}$.	$\therefore \boldsymbol{v} = \boldsymbol{c} + \boldsymbol{d} = \begin{pmatrix}-1\\1\end{pmatrix} + \begin{pmatrix}-3/2\\-3/2\end{pmatrix} = \begin{pmatrix}-5/2\\5/2\end{pmatrix} = -\frac{5}{2}\boldsymbol{i} + \frac{5}{2}\boldsymbol{j}$