Elastic Collisions in One Dimension Cheat Sheet

In this chapter, we will learn to solve problems involving the impact of two particles, where both particles are moving along the same straight line. This is known as a direct impact. There are two main tools to use when tackling questions from this chapter: the principle of conservation of momentum and Newton's law of restitution.

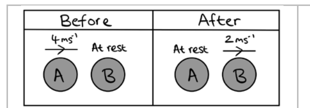
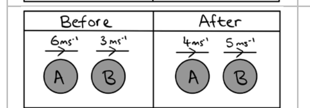
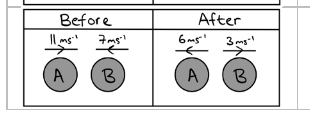
Newton's law of restitution

Newton's law of restitution ($N.L.R$) tells us how the speeds of particles after a collision depend on the material the particles are made from and the speeds with which they collided. Newton's law of restitution states:

- $e = \frac{speed\ of\ separation}{speed\ of\ approach}$ where e is a constant known as the **coefficient of restitution** between the particles.
- e only takes values in the range $0 \le e \le 1$.
  - $e = 1$ represents a perfectly elastic collision where the speed of separation is equal to the speed of approach.
  - $e = 0$ represents a completely inelastic collision where the particles coalesce upon impact (i.e. stick together).

It is important you are confident in finding the speed of separation and speed of approach for any impact. Remember to take into account the direction of motion of each particle. The following example shows how e is calculated in three different scenarios.

Example 1: The three diagrams show the speeds and directions of motion of two particles A and B just before and just after a collision. The particles move on a smooth horizontal plane. Find the coefficient of restitution e in each case.

	a) $speed\ of\ separation = 2 - 0 = 2$ $speed\ of\ approach = 4 - 0 = 4$ $\therefore e = \frac{2}{4} = \frac{1}{2}$
	b) $speed\ of\ separation = 5 - 4 = 1$ $speed\ of\ approach = 6 - 3 = 3$ $\therefore e = \frac{1}{3}$
	c) $speed\ of\ separation = 3 - -6 = 9$ $speed\ of\ approach = 11 - -7 = 18$ $\therefore e = \frac{9}{18} = \frac{1}{2}$

Direct collisions with a smooth plane

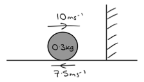
You also need to be able to apply Newton's law of restitution to problems where a particle collides with a fixed smooth plane perpendicular to the direction of motion of the plane. For the direct collision of a particle with a smooth plane, Newton's law of restitution becomes:

- $e = \frac{rebound\ speed}{approach\ speed}$ where e is the coefficient of restitution between the particle and the plane.

An easier way to remember this is as follows:

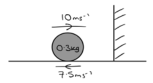
- To find the speed with which a particle rebounds from a fixed plane, multiply the approach speed by e.

Example 2: A small smooth sphere of mass 0.3 kg is moving on a smooth horizontal table with a speed of 10 ms^{-1} when it collides normally with a fixed smooth wall. It rebounds with a speed of 7.5 ms^{-1}. Find the coefficient of restitution between the sphere and the wall.

We start with a diagram.	
Use $rebound\ speed = e \times approach\ speed$	$7.5 = e \times 10$
Solve for e.	$e = \frac{7.5}{10} = 0.75$

Some questions, like the next example, might require you to use $SUVAT$ in addition to $N.L.R$.

Example 3: A particle falls 2.5 m from rest onto a smooth horizontal plane. It then rebounds to a height 1.5 m. Find the coefficient of restitution between the particle and the plane. Give your answer to 2 significant figures.

We start with a diagram.	
Use $SUVAT$ to find the speed of impact.	$u = 0, s = 2.5, a = g$ $v^2 = u^2 + 2as = 5g$ $\therefore v = \sqrt{5g}$
Use $N.L.R.$ to find rebound speed in terms of e.	$rebound\ speed = e \times approach\ speed = e\sqrt{5g}$
Use SUVAT back up until the point of instantaneous rest.	$u = e\sqrt{5g}, v = 0, s = 1.5, a = -g$ $v^2 = u^2 + 2as \Rightarrow 0 = 5ge^2 - 3g$
Solve for e.	$e^2 = \frac{3g}{5g} \Rightarrow e = \sqrt{\frac{3}{5}}$

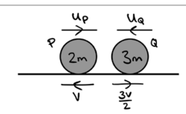
Problems with two unknown velocities

Most exam-style questions will feature a direct impact where two velocities are unknown. For example, you might be given the speeds of two particles before their collision and be expected to find the speeds after the collision. In such cases, you need to use Newton's law of restitution **and** the principle of conservation of momentum (C.L.M). The procedure to find unknown velocities can be summarised into four steps:

1. Form a clear diagram that shows the speeds before the collision as well as the speeds after. It is a good idea to set up your diagrams like we do in Example 4 where the speeds before the collision are labelled on top of the particles and the speeds after labelled below. If you are not told which direction the particles move in after the collision, you can assume any direction as long as it is sensible.
2. Use Newton's law of restitution to form an equation involving e and any velocities.
3. Use the conservation of momentum to form another equation.
4. Solve the two equations formed in steps 2 and 3 simultaneously to find any required velocity.

Of course, not every question will simply ask you to find missing velocities. If the question is worth a lot of marks then it is likely you will need to find any missing velocities before you can completely answer the question, however.

Example 4: Two particles P and Q, of masses $2m$ and $3m$ respectively, are moving in opposite directions along the same straight line on a smooth horizontal plane. The particles collide directly and, as a result of the collision, the direction of motion of P is reversed and the direction of motion of Q is reversed. Immediately after the collision, the speed of P is v and the speed of Q is $\frac{3v}{2}$. The coefficient of restitution between P and Q is $\frac{1}{5}$.
Find: (i) the speed of P immediately before the collision, (ii) the speed of Q immediately before the collision.

We start with a diagram. We take the right direction to be positive.	
We apply the conservation of linear momentum (C.L.M). We label the resultant equation [1].	$2m(u_P) + 3m(-u_Q) = 2m(-v) + 3m\left(\frac{3v}{2}\right)$ $2u_P - 3u_Q = \frac{5v}{2}$ [1]
Apply N.L.R	$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{\frac{3v}{2} + v}{u_P + u_Q} = \frac{\frac{5v}{2}}{u_P + u_Q} = \frac{1}{5}$ $\Rightarrow u_P + u_Q = \frac{25v}{2}$ [2]
Now, we just need to solve equations [1] and [2]. Multiply [2] by 3.	[2] × 3: $3u_P + 3u_Q = \frac{75v}{2}$
Add this equation to [1].	$3u_P + 3u_Q + 2u_P - 3u_Q = \frac{75v}{2} + \frac{5v}{2}$
Simplify and make u_P the subject.	$5u_P = 40v$ $\therefore u_P = 8v$
To find u_Q, substitute u_P back into [1] or [2]. Substituting into [2]:	$u_Q = \frac{25v}{2} - u_P = \frac{25v}{2} - 8v = \frac{9v}{2}$ $\Rightarrow u_Q = \frac{9v}{2}$

Loss of kinetic energy

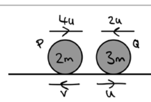
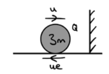
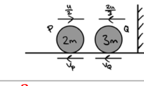
Some questions will ask you to find the loss in kinetic energy as a result of a collision, or even an impulse. You will need to find the total kinetic energy of the system after the collision/impulse and subtract from the total kinetic energy before the collision.

- The loss of kinetic energy is given by $K.E_{before} - K.E_{after}$.

Successive impacts

You may also be asked to solve problems where more than one collision occurs. In such cases, you need to consider each collision separately. The working for each collision is the same as before; you will need to use Newton's law of restitution and/or the principle of conservation of momentum. It is a good idea to set up a different diagram for each collision.

Example 5: Two particles P and Q of masses $2m$ and $3m$ respectively are moving in opposite directions on a smooth plane with speeds $4u$ and $2u$ respectively. The particles collide directly. The direction of motion of Q is reversed by the impact and its speed after impact is u. The particle then hits a smooth vertical wall perpendicular to its direction of motion. The coefficient of restitution between Q and the wall is $\frac{2}{3}$. In the subsequent motion, there is a further collision between Q and P. Find the velocities of P and Q after this collision.

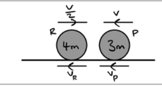
We start with a diagram for the first collision. Take the right direction to be positive.	
We apply the conservation of linear momentum ($C.L.M$).	$2m(4u) + 3m(-2u) = 2m(-v) + 3m(u)$ $-u = 2v$ $\therefore v = -\frac{u}{2}$ Continues →
Now look at the collision with Q and the wall.	
Remember that $rebound\ speed = approach\ speed \times e$	$rebound\ speed = u \times \frac{2}{3} = \frac{2u}{3}$
Now look at the final collision between P and Q.	
Apply $C.L.M.$	$2m\left(\frac{u}{2}\right) + 3m\left(-\frac{2u}{3}\right) = 2m(-v_p) + 3m(-v_Q)$ $-u = -2v_p - 3v_Q$ $u = 2v_p + 3v_Q$ [1]
Apply $N.L.R.$	$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v_p - v_Q}{\frac{u}{2} + \frac{2u}{3}} = \frac{v_p - v_Q}{\frac{7u}{6}}$ $\Rightarrow \frac{7ue}{6} = v_p - v_Q$ [2]
We can solve equations [1] and [2] simultaneously. This gives:	$v_p = \frac{u}{10}(2 + 7e)$ and $v_Q = \frac{u}{15}(3 - 7e)$

Further collision problems

With the help of examples, we will cover two other types of collision problems that tend to crop up in exam questions.

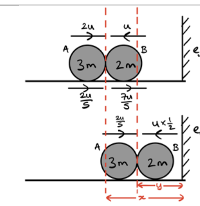
The next example follows on from Example 4.

Example 4, part b: After being struck by Q the particle P collides directly with another particle R, of mass $4m$, which is moving towards P with speed $\frac{v}{2}$. Particles P and R collide directly. The coefficient of restitution between B and C is e.
Given that the direction of motion of B is reversed by this collision, find the range of possible values of e.

We start with a diagram. We take the left direction to be positive.	
We apply the conservation of linear momentum (C.L.M). We label the resultant equation [1].	$4m(0) + 3m(v) = 4m(v_R) + 3m(v_p)$ $3v = 4v_R + 3v_P$ [1]
Apply N.L.R.	$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v_R - v_Q}{v}$ $\Rightarrow ve = v_R - v_Q$ [2]
Now, we just need to solve equations [1] and [2] and find v_P. Doing so gives us:	$v_P = \frac{u}{7}(6e - 5)$
Here is the key part: we assumed that P does not change direction. This means that our expression for v_P is in fact less than zero, because P is travelling in the opposite direction. So, we set $v_P < 0$:	$v_P < 0 \Rightarrow \frac{u}{7}(6e - 5) < 0$ $\Rightarrow 6e - 5 < 0 \therefore e < \frac{5}{6}$
Remember also that $0 \le e \le 1$, so the complete inequality will be:	$0 \le e \quad \therefore \quad 0 \le e < \frac{5}{6}$

In the next example, we will focus on part b. Part a can be solved using the same process as in example 4.

Example 6: A particle A of mass $3m$ is moving in a straight line with speed $2u$ on a smooth horizontal floor. Particle A collides directly with another particle B of mass $2m$ which is moving along the same straight line with speed u but in the opposite direction to A. The coefficient of restitution between A and B is $\frac{1}{3}$. A does not change direction as a result of the collision but B does. (a) Show that the speeds of A and B immediately after the collision are $\frac{2u}{5}$ and $\frac{7u}{5}$ respectively. After the collision, B hits a smooth vertical wall which is perpendicular to the direction of motion of B. The coefficient of restitution between B and the wall is $\frac{1}{2}$. The first collision between A and B occurred at a distance x from the wall. The particles collide again at a distance y from the wall. (b) Find y in terms of x.

We start with two diagrams detailing both collisions. With questions such as this where you want to explicitly relate two distances, the idea is always the same: 1) Consider each particle separately and use $distance = speed \times time$ to find an expression for the total time elapsed between the first and second collisions. Do this for both particles. 2) Use the fact that the total time elapsed is the same for both particles to equate the two equations you already formed, giving you an equation relating the two required distances.	
For A, we use $s = ut$ from first collision to second collision. **By our sketch, we can already spot that the distance between the two collisions is $x - y$.** Let T represent the total time elapsed between the first and second collisions.	$s = x - y$ (from diagram) $u = \frac{2u}{5}, t = T$ $\Rightarrow x - y = \frac{2uT}{5} \therefore T = \frac{5(x-y)}{2u}$ [1]
Now we consider B. B changes speed once it hits the wall, so we need to split the motion up into two parts; let t_1 be the time taken to hit the wall from the first collision and t_2 the time from the wall to the second collision. Remember that when B collides with the wall, $rebound\ speed = e \times approach\ speed$.	$s = ut_1 \Rightarrow x = \frac{7ut_1}{5} \therefore t_1 = \frac{5x}{7u}$ $s = ut_2 \Rightarrow y = \frac{7ut_2}{10} \therefore t_2 = \frac{10y}{7u}$ $T = t_1 + t_2 \Rightarrow T = \frac{5x}{7u} + \frac{10y}{7u}$ [2]
Now, equate [1] and [2].	$\frac{5(x-y)}{2u} = \frac{5x}{7u} + \frac{10y}{7u}$
Multiply both sides by u completely eliminates u. We can then rearrange to make y the subject.	$\frac{5(x-y)}{2} = \frac{5x}{7} + \frac{10y}{7}$ $35x - 35y = 10x + 20y$
The final answer is:	$y = \frac{5}{11}x$