Elastic Strings and Springs Cheat Sheet

Previously, you will have come across many problems involving strings that were modelled as inextensible. That is, where the string does not stretch when it is pulled. We will now learn to solve problems involving a string that is no longer inextensible, but elastic. This means that the string does extend when pulled and will return to its regular length once the pulling force is removed. Problems in this chapter may also involve springs, but the concept is exactly the same. You just need to remember that a compressed spring will produce a thrust (compression force). There are two main concepts that we will cover in this chapter: Hooke's law and elastic potential energy.

Hooke's law

The tension in a stretched elastic string or spring is proportional to its extension. In fact, the tension depends on two things: the natural length of the string/spring and the modulus of elasticity.

- The natural length of a string or spring, l, is the length of the string or spring when it is an unstretched state. Its units are metres (m).
- The modulus of elasticity, λ, is a measure of how "stretchy" the string or spring is. The greater the modulus of elasticity, the less stretchy the string or spring is. Its units are newtons (N).

The following relationship, known as Hooke's law, explicitly defines tension in terms of λ, x and l.

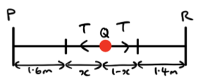
- $T = \frac{\lambda x}{l}$

In reality, Hooke's law only applies for values of x up to a particular value (known as the elastic limit). For this chapter however, you can assume Hooke's law applies for the values given to you in the question.

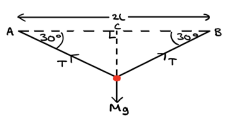
Applying Hooke's law to equilibrium problems

We will now cover two examples showing how we use Hooke's law to solve different types of equilibrium problems. For such questions, it is crucial you are able to firstly draw an accurate diagram). Pay attention to the description in the question and make sure you understand why it leads to the diagram we draw.

Example 1: The elastic springs PQ and QR are joined together at Q to form one long spring. The spring PQ has natural length 1.6 m and modulus of elasticity 20 N. The spring QR has natural length 1.4 m and modulus of elasticity 28 N. The ends, P and R, of the long spring are attached to two fixed points which are 4 m apart. Find the tension in the combined spring.

We start off with a diagram. Note that the tension must be the same in both strings since Q is at rest.	
Looking at the tension in PQ:	$T = \frac{\lambda x}{l} = \frac{20x}{1.6} = 12.5x$
Looking at the tension in QR:	$T = \frac{\lambda x}{l} = \frac{28(1-x)}{1.4} = 20(1-x)$
Equating these expressions for T:	$12.5x = 20(1-x)$
Solving for x:	$12.5x = 20 - 20x$ $\Rightarrow 32.5x = 20$ $\Rightarrow x = \frac{20}{32.5} = \frac{8}{13}$
To find the tension, we can substitute into $x = \frac{8}{13}$ into either of our expressions.	$\therefore T = 12.5\left(\frac{8}{13}\right) = 7.69$ N

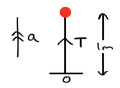
Example 2: An elastic string of natural length $2l$ and modulus of elasticity $4mg$ is stretched between two points A and B. The points A and B are on the same horizontal level and $AB = 2l$. A particle P is attached to the midpoint of the string and hangs in equilibrium with both parts of the string making an angle of $30°$ with the line AB. Find, in terms of m, the mass of the particle.

We start off as usual with a detailed diagram.	
First, let's find the extension for one half of the string and use this with Hooke's law. Use trigonometry with the triangle ACP (h represents the length AP).	$\cos 30 = \frac{adj}{hyp} = \frac{l}{h}$ $\therefore h = \frac{l}{\cos 30} = \frac{2l\sqrt{3}}{3}$
We treat each half of the string separately. Therefore, the extension in one half of the string is $h - l$. The natural length of the complete string is $2l$ so the natural length of half the string will be l.	$x = \frac{2l\sqrt{3}}{3} - l = \frac{-3+2\sqrt{3}}{3}l = 0.155l$
Use Hooke's law.	$T = \frac{\lambda x}{l} = \frac{4mg(0.155l)}{l} = 0.62mg$
Resolve vertically.	$T \sin 30 + T \sin 30 = Mg$
Use $T = 0.62mg$ and make M the subject.	$T = Mg$ $0.62mg = Mg$ $\therefore M = 0.62$ m (2 s.f.)

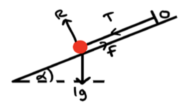
Hooke's law and dynamic problems

You also need to be confident in applying to Hooke's law to questions where a particle is in motion. The principle is the same as in earlier chapters; you will need to use $F = ma$ and/or resolve in a particular direction. The key difference here is that you may also need to use $T = \frac{\lambda x}{l}$. We will go through three examples showing the most common types of dynamics problems where you need to use Hooke's law.

Example 3: A particle of mass 2 kg is attached to one end P of a light elastic spring. The other end Q of the spring is attached to a fixed point O. The spring has natural length 1.5 m and modulus of elasticity 40 N. The particle is held at a point which is 1 m vertically above O and released from rest. Find the initial acceleration of the particle, stating its magnitude and direction.

We start off with a diagram. Note that the spring is **compressed** since the particle is 1 m above O and the natural length is greater (1.5 m). As a result, there is a thrust in the spring which acts upwards.	
Use Hooke's law to find T. Remember that x here represents the compression in the spring.	$T = \frac{\lambda x}{l} = \frac{40(0.5)}{1.5} = \frac{40}{3}$ N
Now use $F = ma$ taking the upwards direction to be positive.	$T - 2g = 2a$ $\frac{40}{3} - 2g = 2a$
Solve for a.	$a = \frac{\frac{40}{3} - 2g}{2} = -3.13\ \text{ms}^{-2}$
State the magnitude and direction.	So, the initial acceleration is $3.13\ \text{ms}^{-2}$ towards O.

Example 4: A particle of mass 1 kg is attached to one end of a light elastic spring of natural length 1.6 m and modulus of elasticity 21.5 N. The other end of the spring is attached to a fixed point O on a rough plane which is inclined to the horizontal at an angle α, where $\tan \alpha = \frac{5}{12}$. The coefficient of friction between the particle and the plane is $\frac{1}{2}$. The particle is held at rest on the plane at a point which is 1.2 m from O down a line of greatest slope of the plane. The particle is released from rest and moves down the slope. A) Find its initial acceleration. B) Without any further calculation, state how your answer to part a would change if the coefficient of friction between the particle and the plane was greater than $\frac{1}{2}$.

a) We start off with a detailed diagram.	
Use Hooke's law to find T. Remember that x here represents the compression in the spring.	$T = \frac{\lambda x}{l} = \frac{21.5(0.4)}{1.6} = 5.375$ N
Now use $F = ma$ down the slope.	$g \sin \alpha + T - F = 1a$
Note that since $\tan \alpha = \frac{5}{12}$, this means thatsin $\alpha = \frac{5}{13}$. You can construct a right-angle triangle to find this. Also, $F = \mu R$ applies here since the particle is in motion so friction is limiting.	$g\left(\frac{5}{13}\right) + 5.375 - \mu R = a$
To find R, resolve perpendicular to the slope.	$R - g \cos \alpha = 0$ $\therefore R = g \cos \alpha = g\left(\frac{12}{13}\right)$
Substitute $R = \frac{12g}{13}$ and $\mu = \frac{1}{2}$ into the equation of motion.	$g\left(\frac{5}{13}\right) + 5.375 - \left(\frac{1}{2}\right)\left(\frac{12g}{13}\right) = a$ $\Rightarrow a = 4.62\ \text{ms}^{-2}$
b) Use the working from part a to put together a coherent explanation.	If the coefficient of friction is greater than $\frac{1}{2}$, then F would be greater. Since $acceleration = g \sin \alpha + T - F$, this means that our acceleration would be smaller in magnitude.

Elastic energy

In Chapter 2 of FM1, you learnt that a particle that is not attached to an elastic string/spring possesses two different types of energy: kinetic and potential. But what if a particle is attached to an elastic string/spring, like in the previous examples? In such cases the particle also possesses another type of energy, known as **elastic potential energy**.

- The elastic potential energy ($E.P.E$) stored in a stretched string/spring is equal to $\frac{\lambda x^2}{2l}$. This is equal to the amount of work done in stretching the string.

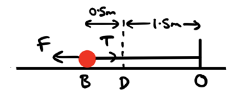
You can also be expected to apply the conservation of energy and work-energy principles to questions involving stretched strings or springs.

- When no external forces other than gravity act upon a particle, the sum of its kinetic energy, gravitational potential energy and elastic potential energy remains constant.
- The change in total energy of a particle is equal to the work done on the particle.

It is important to realise that problems based upon the content in this chapter should be approached in the same way as those in Chapter 2. The only difference is that you may need to use Hooke's law and the consideration of elastic potential energy.

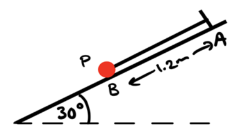
We will now go through two examples of exam-style questions that involve the consideration of energy.

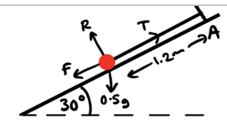
Example 5: A particle P of mass 0.5 kg is attached to one end of a light elastic string of natural length 1.5 m. The other end of the string is attached to a fixed point O on a rough horizontal plane. The coefficient of friction between P and the plane is $\frac{3}{4}$. The particle is held at rest at a point B on the plane, where $OB = 2$ m. When P is at B the tension in the string is 25 N. The particle is then released from rest. The point D is 1.5 m from O and ODB is a straight line. Find the speed of P at the instant when P passes through D.

We start off with a diagram.	
Since friction acts on the particle, we should apply the work-energy principle. We first find the total energy at B. Note that we don't need to consider the GPE since P moves horizontally. Since we are told the tension but not λ, we rewrite $\frac{\lambda x^2}{2l}$ as $T \times \frac{x}{2}$ to find the E.P.E.	$K.E = \frac{1}{2}(0.5)(0)^2$ $E.P.E = \frac{\lambda x^2}{2l} = \frac{\lambda x}{l} \times \frac{x}{2} = T \times \frac{x}{2}$ $= 25 \times \frac{0.5}{2} = 6.25$ J $\therefore$ total energy $= 6.25$ J at B.
Find total energy at D. Note that at D the string is unstretched so $E.P.E = 0$.	$K.E. = \frac{1}{2}(0.5)(v)^2 = 0.25v^2$ $E.P.E = 0$ $\therefore$ total energy $= 0.25v^2$ at D.
Now we find the work done by friction.	$W.D. = Fd = \mu Rd = \left(\frac{3}{4}\right)R(0.5)$
Resolve perpendicular to the plane to find R.	$R - 0.5g = 0$ $\Rightarrow R = 0.5g$
Substitute R into our expression for the work done by friction.	$W.D. = \left(\frac{3}{4}\right)(0.5g)(0.5) = 1.8375$
Now apply the work-energy principle. ($change\ in\ energy = work\ done\ on\ the\ particle$)	$\Rightarrow 6.25 - 0.25v^2 = 1.8375$
Rearrange for v.	$v = \sqrt{\frac{6.25-1.8375}{0.25}} = 4.20\ \text{ms}^{-1}$ to $3\ s.f.$

Example 6: Figure 1 shows a light elastic string, of modulus of elasticity λ newtons and natural length 0.6 m. One end of the string is attached to a fixed point A on a rough plane which is inclined at $30°$ to the horizontal. The other end of the string is attached to a particle P of mass 0.5 kg. The string lies along a line of greatest slope of the plane. The particle is held at rest on the plane at the point B, where B is lower than A and $AB = 1.2$ m. The particle then receives an impulse of magnitude 1.5 Ns in the direction parallel to the string, causing P to move up the plane towards A. The coefficient of friction between P and the plane is 0.7. Given that P comes to rest at the instant when the string becomes slack, find the value of λ.

Figure 1



We start by annotating the diagram.	
The initial impulse is $1.5Ns$ so we use this to find the initial speed.	$I = m(v-u) = 0.5(v-u) = 0.5(u) = 1.5$ $\therefore u = 3\ \text{ms}^{-1} = initial\ speed.$
Find total energy at B. We take B to be the zero level for the $G.P.E$.	$K.E. = \frac{1}{2}(0.5)(3^2) = \frac{9}{4}$ $G.P.E = 0$ $E.P.E = \frac{\lambda x^2}{2l} = \frac{\lambda(0.6)^2}{2(0.6)} = 0.3\lambda$ $\therefore$ total energy $= \frac{9}{4} + 0.3\lambda$ at B.
Find the total energy at point of instantaneous rest. $E.P.E = 0$ since string becomes slack at this point ($x = 0$).	$K.E. = E.P.E = 0$ $G.P.E = mgh = (0.5)(g)(0.6 \sin 30) = 1.47$ $\therefore$ total energy $= 1.47$ J at point of rest.
Find the work done on P by friction.	$W.D. = Fd = \mu Rd = (0.7)(R)(0.6) = 0.42R$
Resolve perpendicular to the plane to find R.	$R - 0.5g \sin 30 = 0$ $\therefore R = 0.5g \sin 30 = 2.45$
Substitute back into our expression for the work done by friction.	$W.D. = 0.42(2.45) = 1.029$ J
Now apply the work-energy principle. ($change\ in\ energy = work\ done\ on\ the\ particle$)	$Change\ in\ energy = energy\ before - energy\ after$ $\Rightarrow \left(\frac{9}{4} + 0.3\lambda\right) - (1.47) = 1.029$
Solve for λ.	$0.3\lambda = 1.002..$ $\Rightarrow \lambda = 3.34$