# Work, Energy and Power II Cheat Sheet



## Work Done by a Variable Force

In reality, there are forces for which their magnitude is not constant. However, integration can be used to find the work done by a force which the magnitude depends on the displacement, $x$ of the object it is acting on. Consider a variable force $f(x)$ acting parallel to the direction of motion. If an object moves in a straight line from a position $x_1$ to a position $x_2$ under the action of this variable force $f(x)$, the work done can be calculated using the following formula:

$$\text{Work Done} = \int_{x_1}^{x_2} f(x)\,dx.$$

Example 1: The work done by a force $kx^3$ in displacing an object from $x = 0$ to $x = 4$ is 192 J. Find the value of $k$.

| | |
|---|---|
| Set up an equation for the work done using the formula above. | $\int_0^4 kx^3\,dx = 192$ |
| Integrate and substitute limits in to solve for $k$. | $192 = \left[\frac{kx^4}{4}\right]_0^4 = 64k \Rightarrow k = 3\text{Nm}^{-3}$ |

Example 2: A bus of mass 3000 kg begins stationary at point A at $x = 0$. The bus then begins to move under a constant driving force of 3600 N. The bus also experiences a resistive force of $\frac{x^2}{16}$ N. The bus moves along a smooth horizontal road and passes by point B, 200 m away. Find the speed of the bus as it passes point B.

| | |
|---|---|
| Calculate the work done by the driving force between points A and B. | Work done by driving force $= 3600 \times 200 = 720{,}000$ J |
| Calculate the work done against resistance between points A and B. | Work done against resistance $= \int_0^{200} \frac{x^2}{16}dx = \left[\frac{x^3}{48}\right]_0^{200} = \frac{500000}{3}$ J |
| Use conservation of energy to set up an equation for increase in kinetic energy. Work done by driving force – work done against resistance = increase in kinetic energy. Then rearrange to solve for $v$. | $720{,}000 - \frac{500000}{3} = \frac{1}{2} \times 3000 \times v^2$ $\Rightarrow v^2 = \frac{3320}{9}$ $\Rightarrow \lvert v \rvert = 19.2\ \text{ms}^{-1}\ (3s.f.)$ |

## Hooke's Law and the Modulus of Elasticity

When elastic springs are compressed or stretched, they exert a restoring force to bring the length of the spring back to its natural length (the length of the spring when it is not acted on by any tension or compression forces). When the spring is extended, the restorative force will be tension. Conversely, when the spring is compressed, the restorative force is thrust. A similar restorative force acts in elastic strings, however elastic strings can only be extended and not compressed. While the displacement from the natural length is within the string or spring's elastic limit, this restorative force can be described by Hooke's Law which states:

$$T = kx.$$

In words, the restorative force $T$ exerted by a spring is proportional to the extension $x$ acting in the opposite direction to the extension. The constant $k$ is known as the spring constant or stiffness. An alternative formulation of Hooke's Law which uses a different constant of proportionality is

$$T = \frac{\lambda}{l}x,$$

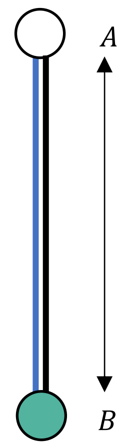
where $l$ is the natural length of the spring or string and $\lambda$ is the modulus of elasticity. The modulus of elasticity is the force required to double the length of an elastic spring or string, so a low value means the spring or string is quite flexible and easy to compress. A well-known example of an elastic modulus is the Young's modulus, however this is not the only one. An assumption required for Hooke's Law to hold is that the spring or string is light, this ensures the restorative force (tension or thrust) does not vary along the length of the spring or string. Hooke's Law can be applied to solve many problems involving strings or springs.

Example 3: A light elastic spring with modulus of elasticity 80 N lies flat on a horizontal table fixed at one end, it is compressed horizontally with a force of 70 N. If the spring has a natural length of 2 m find the change in length.

| | |
|---|---|
| Write Hooke's Law | $T = \frac{\lambda x}{l}$ |
| Rearrange to make $x$ the subject and substitute values in to obtain $x$. | $x = \frac{Tl}{\lambda} = \frac{70 \times 2}{80} = 1.75\text{m}$ |

Example 4: Two strings hang in equilibrium with their top ends vertically suspended from a fixed point A, and their bottom ends attached to an object B, vertically below A as shown in the diagram. Object B has mass 2 kg. The blue string (string 1) has a natural length of 1 m and stiffness of 6 Nm$^{-1}$, and the black string (string 2) has a natural length of 1.2 m and a stiffness of 10 Nm$^{-1}$. If the distance between $A$ and $B$ is $D$, find $D$.

| | |
|---|---|
| System at equilibrium so there is zero resultant force acting on $B$. Therefore, the sum of the tensions in the strings is equal to the weight of the object. | $T_1 + T_2 - 2g = 0$ |
| Use Hooke's Law to find $T_1$ and $T_2$. | $T_1 = 6(D-1),\ T_2 = 10(D-1.2)$ |
| Substitute $T_1$ and $T_2$ into the equilibrium equation. | $6(D-1) + 10(D-1.2) = 2g$ |
| Rearrange to solve for $D$. | $16D - 6 - 12 = 2g \Rightarrow \text{D} = 2.35\text{ m}$ |




## Elastic Potential Energy

For a light elastic spring or string to be extended, work must be done; this work done is stored as elastic potential energy. After the spring or string is released, elastic potential energy is converted into kinetic energy as the spring or string contracts back to its natural length. This is analogous in the compression of a light elastic spring. This work done extending an elastic spring or string can be calculated using Hooke's Law and the formula for work done by a variable force. The work done in extending a spring or string from extension $x_1$ to extension $x_2$ is given by:

$$\text{Work done} = \int_{x_1}^{x_2} T\,dx = \int_{x_1}^{x_2} \frac{\lambda x}{l}\,dx,$$

Integrating and substituting the limits in:

$$\text{Work done} = \left[\frac{\lambda x^2}{2l}\right]_{x_1}^{x_2} = \frac{\lambda}{2l}(x_2^2 - x_1^2)$$

Or equivalently:

$$\text{Work done} = \frac{k}{2}(x_2^2 - x_1^2)$$

This formula also holds for the compression of a light elastic spring, from compression $x_1$ to compression $x_2$ and can be derived using the same method.

This formula for work done in extending or compressing a light elastic spring or string can be used to find a formula for the elastic potential energy ($EPE$) stored. This is done by setting $x_1 = 0$, so extension or compression is from the natural length. Hence the following formulae are obtained:

$$EPE = \frac{kx^2}{2} = \frac{\lambda x^2}{2l}$$

This formula is often used when applying the principle of conservation of energy. When an object is acted on only by its weight and the force in a light elastic string or spring:

$$GPE + EPE + KE = \text{constant}$$

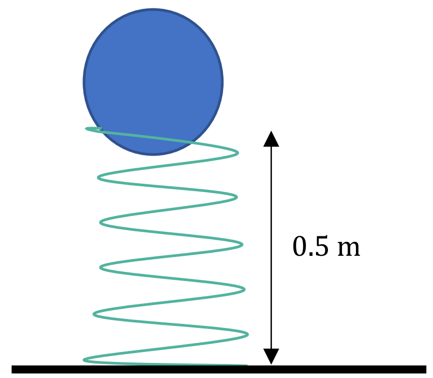
where $GPE$ is gravitational potential energy, $EPE$ is elastic potential energy and $KE$ is kinetic energy.

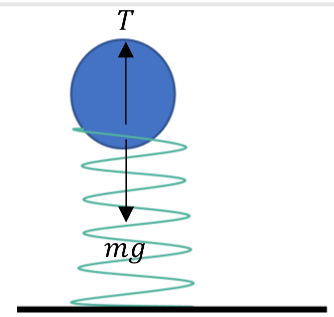
Example 5: A ball with mass 2 kg is attached on top of a spring with natural length 0.7 m, resting on a horizontal table. Initially the system is at equilibrium with the ball 0.5 m above the table, as in shown in the diagram.

a) Calculate the modulus of elasticity $\lambda$ of the spring.

The spring is then compressed a further 0.1 m

b) Using conservation of energy, find the maximum speed of the ball in its subsequent vertical motion.




| | |
|---|---|
| a) Draw a diagram with forces labelled at equilibrium |   |
| Use Hooke's Law to set up equation at equilibrium and substitute values in | $\frac{\lambda x}{l} - mg = 0$ $\Rightarrow \lambda = \frac{mgl}{x} = \frac{2g \times 0.7}{0.7 - 0.5} = \frac{343}{5} = 68.6\text{ N}$ |
| b) Use conservation of energy, i.e. $EPE_1 + GPE_1 + KE_1 = EPE_2 + GPE_2 + KE_2$. | Set GPE at start of upward movement to 0 so $GPE_1 = 0$. Ball initially at rest so $KE_1 = 0$. So $EPE_1 = GPE_2 + KE_2 + EPE_2\ (*)$ |
| Find the initial compression by adding the additional compression to the equilibrium compression | Initial compression $= (0.7 - 0.5) + 0.1 = 0.3$ |
| Substitute into $(*)$ to relate the final compression of the spring $x$ to the speed of the ball $v$. Observe that the distance moved by the ball $h$ = initial compression – final compression $= 0.3 - x$ | $\frac{68.6}{2 \times 0.7}(0.3)^2 = 2g(0.3 - x) + \frac{1}{2} \times 2v^2 + \frac{68.6}{2 \times 0.7}x^2$ $\Rightarrow v^2 = 4.41 - 0.6g + 2gx - 49x^2$ |
| Use $g = 9.8\text{ ms}^{-2}$ to simplify and then complete square of quadratic to find maximum $v$. Notice that the maximum speed occurs when the compression is at the equilibrium compression. | $v^2 = -49(x^2 - 0.4x) + 4.41 - 5.88$ $\Rightarrow v^2 = -49(x - 0.4)^2 + 0.49$ $\Rightarrow v_{\max}^2 = 0.49$ $v_{\max} = 0.7\text{ms}^{-1}$ |