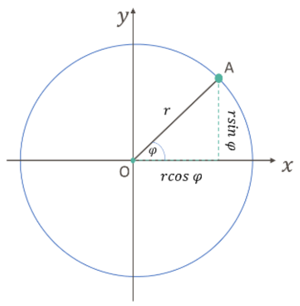
Circular Motion II Cheat Sheet (A Level Only)

AQA A Level Further Maths: Mechanics

Kinetic Quantities as Vectors in Circular Motion



A particle moving in a horizontal circular orbit can have its displacement, velocity and acceleration described by vectors. Consider a particle moving in a horizontal circular path in the xy-plane, as seen in the diagram on the left. The position of the particle has both x and y coordinates, so is written as the position vector in terms of the angle φ measured anticlockwise from OA or the constant angular speed ω and time, t:

$$\boldsymbol{r} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix} = \begin{pmatrix} r\cos\omega t \\ r\sin\omega t \end{pmatrix}$$

To find the velocity vector for the particle, differentiate the position vector with respect to time and repeat the method to find the acceleration vector. The chain rule needs to be used to complete the differentiation.

The velocity vector is given by $\boldsymbol{v} = \dot{\boldsymbol{r}} = \begin{pmatrix} -\omega r\sin\omega t \\ \omega r\cos\omega t \end{pmatrix}$. Its direction is along the tangent through point A.

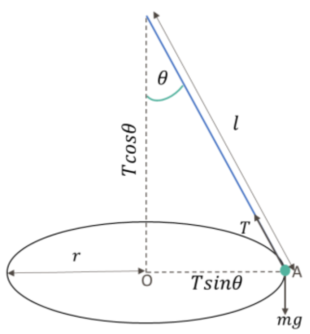
The acceleration vector is $\boldsymbol{a} = \begin{pmatrix} -\omega^2 r\cos\omega t \\ -\omega^2 r\sin\omega t \end{pmatrix} = -\omega^2\boldsymbol{r}$. It acts towards the centre of the circular path.

Example 1: A planet of mass 10^{10}kg is in horizontal circular motion around a star and has a position vector given by $\begin{pmatrix} 3\cos\pi t + 2 \\ 3\sin\pi t - 1 \end{pmatrix}$ km. You may assume the planet moves in an xy-plane.

a) Find the linear speed of the particle.
b) Find the force which keeps the planet in its circular orbit. Give your final answer in standard form.

a) Differentiating the position vector with respect to time gives the position vector.	$\boldsymbol{v} = \begin{pmatrix} \frac{d}{dt}(3\cos\pi t + 2) \\ \frac{d}{dt}(3\sin\pi t - 1) \end{pmatrix} = \begin{pmatrix} -3\pi\sin\pi t \\ 3\pi\cos\pi t \end{pmatrix}$
By finding the modulus of the velocity, the linear speed is calculated. Recall the trigonometric identify: $\sin^2 x + \cos^2 x = 1$. This is used to simplify the equation.	$\lvert\boldsymbol{v}\rvert = \sqrt{(-3\pi\sin\pi t)^2 + (3\pi\cos\pi t)^2}$ $= \sqrt{9\pi^2(\sin^2\pi t + \cos^2\pi t)}$ $= 9.4\ \text{ms}^{-1}$
b) To find the force, you can use Newton's second law: $F = m\lvert\boldsymbol{a}\rvert$. Differentiating the velocity vector with respect to time allows you to calculate the acceleration vector to use in the equation and then to find the modulus to substitute into the equation. Make sure you give your final answer in standard form as requested.	$\boldsymbol{a} = \begin{pmatrix} \frac{d}{dt}(-3\pi\sin\pi t) \\ \frac{d}{dt}(3\pi\cos\pi t) \end{pmatrix} = \begin{pmatrix} -3\pi^2\cos\pi t \\ -3\pi^2\sin\pi t \end{pmatrix}$ $\lvert\boldsymbol{a}\rvert = \sqrt{(-3\pi^2\cos\pi t)^2 + (3\pi^2\sin\pi t)^2}$ $= \sqrt{9\pi^4(\cos^2\pi t + \sin^2\pi t)}$ $= \sqrt{9\pi^4} = 3\pi^2\ \text{ms}^{-2}$ Using $F = m\lvert\boldsymbol{a}\rvert$: $F = 3\pi^2 \times 10^{10}\ N = 3.0 \times 10^{10}$ N (2s.f.)

Conical Pendulum



The motion of a conical pendulum is an example of three-dimensional horizontal circular motion. It consists of simple pendulum with a string or rod of length l attached to a fixed pivot with a bob/weight of mass m at the other end of the string or rod. The bob can be imagined as moving round in a circle at a constant angular speed ω while the string or rod traces out the cone shape creating an angle θ between the string and the vertical component.

Examples of conical pendulums can consist of one string, as seen in the diagram, or two strings. To solve conical problems, the forces parallel and perpendicular to the plane of circular motion need to be resolved. The parallel resultant forces are equivalent to the centripetal force $F = \frac{mv^2}{r} = m\omega^2 r$. The relationship can be written as:

$$F\sin\theta = \frac{mv^2}{R}$$

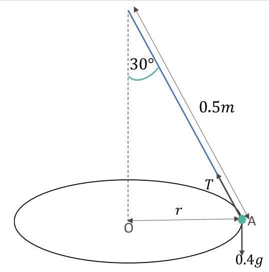
The perpendicular resultant forces are contributed to by the weight of the bob, mg, and vertical component of tension, $F\cos\theta$. These forces oppose each other and thus the relationship is written as:

$$F\cos\theta = mg$$

One String

Example 2: A garden game consists of a 0.5m long, light, and inextensible string attached to a fixed-point A on one end and to a ball of mass 0.4kg at the other. When the ball is it hit, the curved surface of a cone is drawn out as the ball moves in a horizontal circular path and make an angle of 30^0 with the downwards vertical. Using $g = 9.8\ \text{ms}^{-2}$, find:

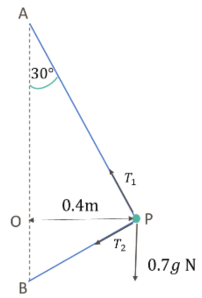
a) The tension in the string.
b) The angular speed of the particle.

a) To begin you can draw out a diagram and label it with the information that has been given in the question. The particle is moving horizontally. In this diagram, T is the tension in the string and r is the radius of the circle.	
To find the tension, you need to resolve the forces. In this case, resolving vertically or perpendicularly to the plane of the motion is best because you have m and θ.	$T\cos\theta - mg = 0$ $T\cos 30° - 0.4g = 0$ $T = 4.5$ N (2s.f.)
b) To find the angular speed, begin by resolving horizontally or parallel to the plane of motion. This allows the radial acceleration to be found.	$T\sin\theta = ma$ $4.53\sin 30° = 0.4a$ $a = \frac{4.53\sin 30°}{0.4} = 5.7\ \text{ms}^{-2}$ (2s.f.)
The radius of the circular motion is found by using ℓ, the length of the string, and trigonometry.	$r = \ell\sin\theta = 0.5\sin 30° = 0.25$ m
Recall that the acceleration of a particle in circular motion is $a = \omega^2 r$. It is then possible to use the values of r and ω to substitute into the acceleration equation.	$a = \omega^2 r$ Rearrange for ω: $\omega = \sqrt{\frac{a}{r}} = = \sqrt{\frac{5.7}{0.25}} = 4.8\ \text{ms}^{-1}$ (2s.f.)

Two Strings

Example 3: A bead P of mass 700g is on a light inextensible string and positioned at point P as shown in the diagram. The other two ends of the strings are fixed at A and B. The particle rotates in a horizontal circular path of radius 0.4m when both strings are taut and at a constant angular speed of 0.3 rad s-1. The angle OAB is 30° and at P the two ends of the string form a right angle. Calculate:

a) The tension in the string.
b) The radial acceleration of the bead.



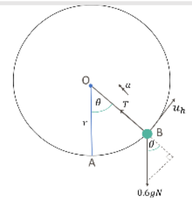
a) The horizontal and vertical forces at P need to be resolved to set up equations of motions to find the tension. The tension equations can then be solved as simultaneous equations. The total tension of the string is the sum of the two components. Do not forget to convert the values into the correct units in particular the mass using the conversion 1kg = 1000g.	Resolving horizontally: $T_1\sin 30° = T_2\sin 60°$ $T_1 = \frac{\sin 60°}{\sin 30°}T_2 = \sqrt{3}T_2$ Resolving vertically: $T_1\cos 30° = T_2\cos 60° + 0.7g$ $\frac{\sqrt{3}}{2}T_1 = \frac{1}{2}T_2 + 0.7g$ The two equations can be solved simultaneously: $\frac{\sqrt{3}}{2}(\sqrt{3}T_2) = \frac{1}{2}T_2 + 0.7g$ $T_2 = 0.7g$ N $T_1 = \sqrt{3}\,T_2 = 0.7g\sqrt{3}$ N The total tension is therefore: $T = T_1 + T_2 = 0.7g(1 + \sqrt{3})$ N $= 18.7$ N (3s.f.)
b) The radial acceleration of the particle can be found using the equation: $a = \omega^2 r$.	$a = \omega^2 r = 0.3^2 \times 0.4 = 0.036\ \text{ms}^{-2}$

Circular Motion in a Vertical Plane

Particles moving in a circular path could also have a variable speed. In this case, particles will have a different initial velocity, u, and final velocity, v. For a particle in vertical circular motion, the principle of conservation of mechanical energy can be used to determine the speed of the particle at a point in its motion. It is important to remember that the centripetal force which a particle in vertical circular motion experiences results from a combination of gravity and tension (not just tension, as in the horizontal case). The radial acceleration acts in the same direction as the centripetal force: towards the centre of the circle. When a particle is attached to a light inextensible string while moving in a vertical circle, the tension in the string is greater than zero when the particle is at the highest point of the orbit. For a particle attached to a light rod, the particle's speed is greater than zero when it is at the highest point of the circular path.

Example 4: Helen is holding a yo-yo of 0.6 kg attached to one end of a 0.7 m long light inextensible string. Helen lets the yo-yo hang in equilibrium at its lowest point. She begins to swing the yo-yo upwards projecting it upwards at a horizontal speed of u_h ms-1. You may assume the end of the string held by Helen is a fixed point and that the yo-yo is free to rotate in a vertical circle.

a) Find an expression for the tenson in the string when it makes an angle of θ with the downwards vertical going through the centre of the circle.
b) Find out the values of u_h for which the yo-yo will make a complete circle.

a) It is good to begin by drawing out a diagram to identify the motion of the yo-yo. It is important to label the diagram with key information. The radial component of acceleration is labelled as a in this diagram.	 At A, the gravitational potential energy is zero. The yo-yo is at position B on its circular path.
It is essential to consider the energy conservation of the system for vertical circular motion. The yo-yo forces are labelled on the diagram where T is the tension in the string. The gravitational potential energy and kinetic energy of the yo-yo can be considered when it is at position A and B. Due to the energy conservation principle, the two energy equations at A and B can be equated, and you can assume no energy is lost as the yo-yo moves.	At A: $E_{GP} + E_K = 0 + \frac{1}{2}mE_K = \frac{1}{2}(0.6)(u_h)^2 = 0.3u_h^2$ At B: Due to energy conservation: $E_{GP} + E_K = mgh + \frac{1}{2}mu_B^2$ $= mgr(1-\cos\theta) + \frac{1}{2}mu_B^2$ $= 0.6g \times 0.7(1-\cos\theta) + \frac{1}{2}(0.6)(u_B)^2$ $= 0.42g(1-\cos\theta) + 0.3u_B^2$ $0.3u_h^2 = 0.42g(1-\cos\theta) + 0.3u_B^2$ $u_B^2 = u_h^2 - 1.4g(1-\cos\theta)$
Remember the formula for radial acceleration is $a = \frac{v^2}{r}$. Newton's second law, $F = ma$, can be used in conjunction with resolving the forces in the radial direction to find an equation for motion for the yo-yo. In this case the equation is $T - W = ma$ where W is the weight of the yo-yo in the radial direction. Then resolve the equation find an expression for T.	$a = \frac{v^2}{r} = \frac{u_h^2 - 1.4g(1-\cos\theta)}{0.7}$ Using Newton's second law, $F = ma$: $T - 0.6g\cos\theta = 0.6\,\frac{u_h^2 - 1.4g(1-\cos\theta)}{0.7}$ Rearrange for T: $T = \frac{6}{7}u_h^2 - 2g - \frac{1}{6}g\cos\theta$
b) When the yo-yo is at the highest point of the circular path, the angle $\theta = 180°$ with respect to the downwards vertical. The speed of the yo-yo must also be large enough that the string maintains some tension, so the string is taut when the yo-yo is at the highest point of the circular path. You can use this to rearrange the inequality to find an expression for the range of values u_h can take. In this case it is a lower bound for u_h.	For a full circle, $T > 0$ when $\theta = 180°$: $\frac{6}{7}u_h^2 - 2g - \frac{1}{6}g\cos 180 > 0$ $\frac{6}{7}u_h^2 - 2g + \frac{1}{6}g > 0$ $\frac{6}{7}u_h^2 > \frac{11}{6}g$ $u_h > \sqrt{\frac{77}{36}g}$ $u_h > 4.6\ \text{ms}^{-1}$ (2s.f.) For the yo-yo to complete a full circle $u_h > 4.6\ \text{ms}^{-1}$.