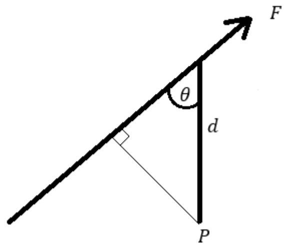
Centre of Mass and Moments VI Cheat Sheet

AQA A Level Further Maths: Mechanics

Forces Acting on a Rigid Body in Equilibrium Using Moments

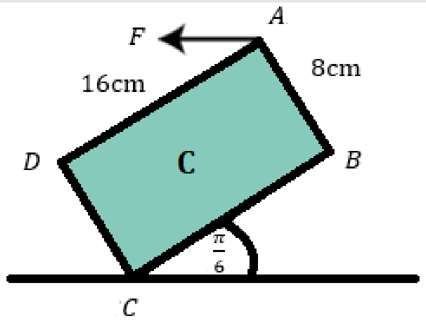
Rigid bodies, such as laminae or rods, may be held in equilibrium by a force, or forces, acting against their natural rotation when suspended. To find the magnitudes and directions of such forces, moments must be considered about a point within the lamina. Recall that the moment of a force F about a point P is given by $Fd\sin(\theta)$ where d is the distance between the line of action of the force F and the point P, and θ is the angle as shown in the diagram to the right.



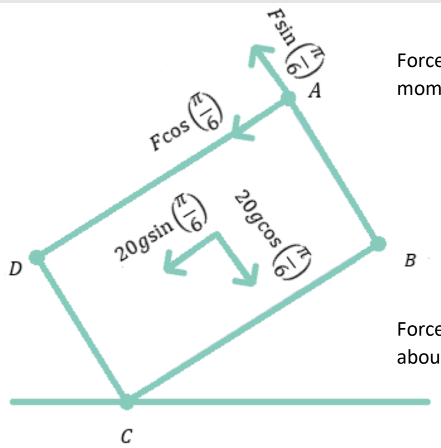
The results are independent of the point chosen, and so choosing a point through which some forces act can help to simplify the problem.

Example 1: A uniform rectangular lamina $ABCD$ of mass 20kg is held in equilibrium by a horizontal force F applied at vertex A. Vertex C rests on the ground. Given that $AB = 8\text{cm}$, $BC = 16\text{cm}$, and the side BC forms an angle of $\frac{\pi}{6}$ radians with the horizontal, find the magnitude of F, using $g = 9.8\text{ms}^{-2}$.

Begin by drawing a diagram of the lamina, including any measurements and angles.



Draw the components of the known forces onto the diagram. Since the reaction force at vertex C is unknown, take moments about this point. This means that it will not have to be considered. The angle between F and the edge is the same as between edge BC and the horizontal since AD and BC are parallel.



Forces causing anticlockwise moments about C:

1) $20g\sin\left(\frac{\pi}{6}\right)$
2) $F\sin\left(\frac{\pi}{6}\right)$
3) $F\cos\left(\frac{\pi}{6}\right)$

Force causing clockwise moments about C:

$$20g\cos\left(\frac{\pi}{6}\right)$$

Since the object is in equilibrium, the sum of the clockwise moments is equal to the sum of the anticlockwise moments. Find expressions for both by multiplying the relevant forces by their distance from C. Recall that the centre of mass will be in the geometric centre since the lamina is uniform.

Anticlockwise moments:

$$\left(20g\sin\left(\frac{\pi}{6}\right)\cdot 0.04\right) + \left(F\sin\left(\frac{\pi}{6}\right)\cdot 0.16\right) + \left(F\cos\left(\frac{\pi}{6}\right)\cdot 0.08\right)$$

$$= \frac{2g}{5} + \left(\frac{2+\sqrt{3}}{25}\right)F$$

Clockwise moments:

$$20g\cos\left(\frac{\pi}{6}\right)\cdot 0.08 = \frac{4\sqrt{3}g}{5}$$

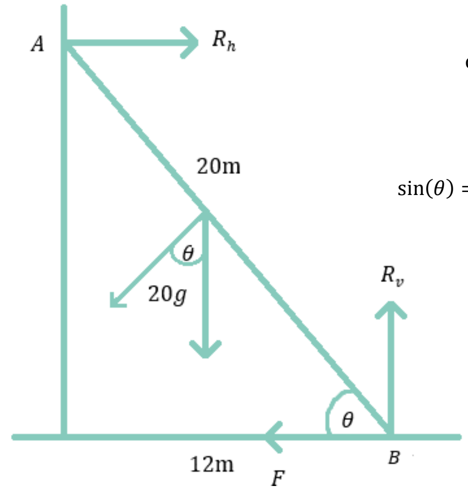
Equate the clockwise and anticlockwise moments to form an equation in F and solve.

$$\frac{2g}{5} + \left(\frac{2+\sqrt{3}}{25}\right)F = \frac{4\sqrt{3}}{5}g$$

$$\Rightarrow F = \frac{25g}{2+\sqrt{3}}\left(\frac{4\sqrt{3}}{5} - \frac{2}{5}\right) = 65\text{N}$$

Example 2: A uniform ladder AB of mass 20kg and length 20m rests with one end, B, on rough horizontal ground, and the other, A, against a smooth wall. The foot of the ladder is 12m from the wall. Given that the ladder is on the point of slipping, find a) the coefficient of friction, μ, between the ladder and the ground and b) the reaction force at the point A, using $g = 9.8\text{ms}^{-1}$.

a). Begin by drawing a diagram of the system, including all the forces and lengths. There is a horizontal reaction force at the point A, shown here as R_h, and a vertical reaction force at the point B, denoted by R_v. The frictional force is denoted by F. Use Pythagoras's theorem to find the sine and cosine of the unknown angle, θ, ready to be used in later calculations.



$$\cos(\theta) = \frac{12}{20} = \frac{3}{5}$$

$$\sin(\theta) = \frac{\sqrt{20^2 - 12^2}}{20} = \frac{16}{20} = \frac{4}{5}$$

Since the ladder is in equilibrium, there is no resultant force in any direction. Use this to find the value of R_v. Furthermore, since the ladder is on the point of slipping, the frictional force is at its maximum. Use this and R_v to find F.

Upwards and downwards forces are equal:

$$R_v - 20g = 0 \Rightarrow R_v = 20g$$

Maximal frictional force:

$$F = \mu R_v = 20\mu g$$

The ladder is on the point of slipping about the point B. Take moments about A, revoking the need to find R_h. Use that centre of mass of the rod is at its midpoint, since it is uniform, to find the moment of the weight about A.

Clockwise moments about A:

$$(20g\cos(\theta)\cdot 10) + (20\mu g\cdot\sin(\theta)\cdot 20) = 120g + 320\mu g$$

Anticlockwise moments about A:

$$R_v\sin\left(\frac{\pi}{2} - \theta\right)\cdot 20 = 20R_v\cos(\theta) = 20\cdot 20g\cdot\frac{3}{5} = 240g$$

Equating the clockwise and anticlockwise moments gives:

$$120g + 320\mu g = 240g$$

$$\therefore \mu = \frac{(240g - 120g)}{320g} = \frac{3}{8}$$

b). Use that the ladder is in equilibrium, meaning that there is no resultant horizontal force, to find R_h.

Forces acting to the left are equal to the forces acting to the right:

$$R_h = F = 20\mu g = 20\cdot\frac{3}{8}\cdot 9.8 = 74\text{N}$$

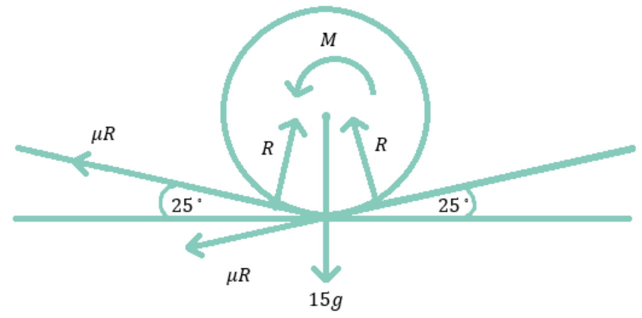
Forces Acting on a Rigid Body in Equilibrium Using Couples

Recall that two forces equal in magnitude and acting in opposite directions along different lines of action have a turning effect, despite having zero resultant force. Forces that fit this description form a couple and cause a rigid body to rotate.

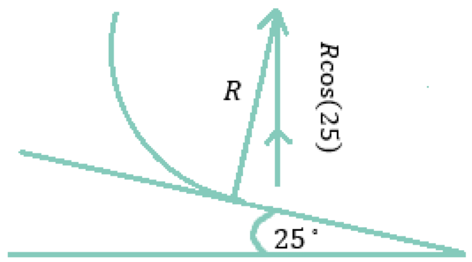
Rigid bodies on rough surfaces can withstand some rotational force without moving. The magnitude of the rotational force they can resist depends on the coefficient of friction, i.e., the roughness, of the surface they are on.

Example 3: A uniform cylinder of radius 2m and mass 15kg rests in equilibrium with its curved surface in contact with two rough planes, each with coefficient of friction $\mu = \frac{3}{7}$, and each inclined at an angle of 25˚ to the horizontal. The line of intersection of the axis of the cylinder and the intersection of the planes is horizontal. Find the magnitude of the maximum couple, M, that can be applied to the cylinder, in the plane perpendicular to its axis, such that the cylinder remains in equilibrium. Take $g = 9.8\text{ms}^{-1}$.

Draw a diagram of the system, including all forces acting. Choose a direction for the cylinder to rotate in and direct the friction against this direction of rotation. M here is the magnitude of the couple, and R the reaction force between the cylinder and the planes. $15g$ is the cylinder's weight.



Resolve the reaction forces into their vertical components. The angle between R and the vertical is the same as between the plane and the horizontal, 25˚. Find its vertical component by using $R\cos(25)$. This is the same for both reaction forces, by symmetry.



Since the cylinder is in equilibrium, there is no resultant force upwards. Equate the upwards and downwards forces to find R. To find the maximum couple, equate the anticlockwise moments about the centre of the cylinder to M, since friction will act in the opposite direction to the rotation. Use the maximum frictional force. Friction acts where the cylinder is in contact with the plane, so 2m from the centre.

Resolving forces perpendicular to the horizontal (vertically upwards):

$$15g + \mu R\sin(25) = \mu R\sin(25) + R\cos(25) + R\cos(25)$$

$$\therefore 15g = 2R\cos(25)$$

$$\therefore R = \frac{15g}{2\cos(25)}\text{N}$$

Taking moments about the centre of the cylinder:

$$2(2\cdot\mu R) - M = 0$$

$$\therefore M = 4\mu R = 4\cdot\left(\frac{3}{7}\right)\cdot\frac{15g}{2\cos(25)} = 140\text{Nm}$$