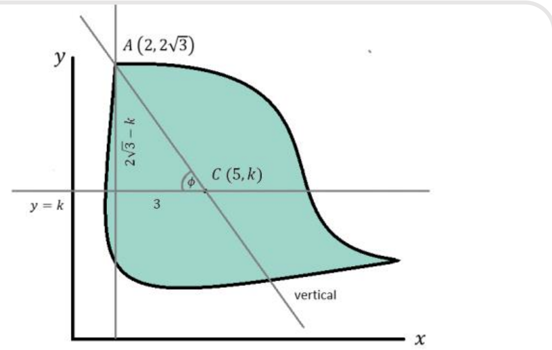
Centre of Mass and Moments V (A Level Only)

Suspension of Rigid Bodies from a Point

Rigid bodies are objects that are fixed in shape, and consist of combinations of laminas, wires, rods, solids, or particles. They can be either a single entity, or a composite body. When a rigid body is suspended from a point, it will hang with its centre of mass vertically below the suspension point. This is so that the object's weight does not have any turning effect on itself, and so it is in equilibrium.

Example 1: A uniform lamina of mass M has its centre of mass at the point $C = (5, k)$ and is freely suspended from the point A where $A = (2, 2\sqrt{3})$. When the lamina is hung from A, the angle, ϕ, between the vertical and the line $y = k$ is $\frac{\pi}{6}$ radians. Find the value of k.

Draw lines onto a diagram of the lamina to help with visualising the problem. When suspended from A, the vertical will pass through the centre of mass. Create a triangle which includes the angle ϕ and write on the unknown lengths.



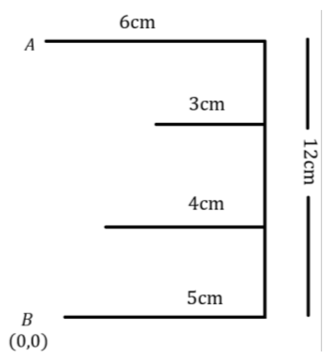
Use the fact that $\tan(\phi)$ is equal to the ratio of the opposite and adjacent side to set up an equation for k and then solve.

$$\phi = \frac{\pi}{6} \Rightarrow \tan(\phi) = \frac{\sqrt{3}}{3}$$

$$\Rightarrow \frac{2\sqrt{3} - k}{3} = \frac{\sqrt{3}}{3} \Rightarrow k = \sqrt{3}$$

Example 2: A composite body is comprised of several pieces of uniform wire, with lengths totalling 30cm. The horizontal wires are equal spaced along the vertical wire. See below for their formation. Let A be the left-hand end of the 6cm wire and the point B be the point vertically below it, in line with the 5cm piece of wire. The body is hung from the point A. Find the angle ϕ between the line AB and the vertical to three significant figures.

Find the centres of mass of each piece of wire individually, relative to the point B. Use that the centre of mass of a uniform piece of wire is at its midpoint, and that the vertical distance from one piece of horizontal wire to the next is 4cm.



Centres of mass of components:

6cm Wire: $\binom{3}{12}$

3cm Wire: $\binom{6-\frac{3}{2}}{12-4} = \binom{\frac{9}{2}}{8}$

4cm Wire: $\binom{6-\frac{4}{2}}{12-(4+4)} = \binom{4}{4}$

5cm Wire: $\binom{6-\frac{5}{2}}{12-(4+4+4)} = \binom{\frac{7}{2}}{0}$

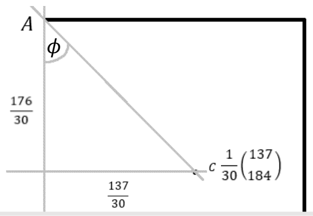
12cm Wire: $\binom{6}{6}$

Use the formula for the centre of mass of a composite body to find the centre of mass. The wires' lengths can be used in place of their masses since they are uniform, meaning mass and length are proportional.

$$30\binom{\bar{x}}{\bar{y}} = 6\binom{3}{12} + 3\binom{\frac{9}{2}}{8} + 4\binom{4}{4} + 5\binom{\frac{7}{2}}{0} + 12\binom{6}{6} = \binom{137}{184}$$

$$\therefore \binom{\bar{x}}{\bar{y}} = \frac{1}{30}\binom{137}{184}$$

The vertical will pass through the centre of mass. Create a right-angled triangle, with side lengths of $\frac{137}{30}$ and $12 - \frac{184}{30} = \frac{176}{30}$. Use that $\tan(\phi)$ will be equal to the ratio of the opposite $\left(\frac{137}{30}\text{cm}\right)$ side to the adjacent $\left(\frac{176}{30}\text{cm}\right)$ side to find ϕ.

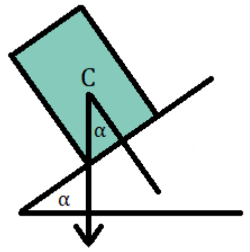


$$\tan(\phi) = \frac{\frac{137}{30}}{\frac{176}{30}} = \frac{137}{176}$$

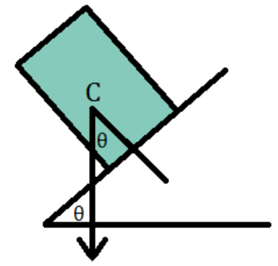
$$\Rightarrow \phi = \tan^{-1}\left(\frac{137}{176}\right) = 0.661 \text{ rad}$$

Conditions for a Lamina to Topple on an Inclined Plane

Consider a lamina placed on a rough inclined plane. If there is sufficient friction to prevent the lamina from sliding down the plane, it may topple. A lamina is in stable equilibrium if a vertical line drawn from its centre of mass lies within the line of contact of the base with the plane. It will not topple in this case. If the centre of mass is directly above the end of the line of contact with the plane, the lamina is in limiting equilibrium and is *about* to topple. The lamina topples the moment that the vertical line though its centre of mass falls outside of its line of contact with the plane.

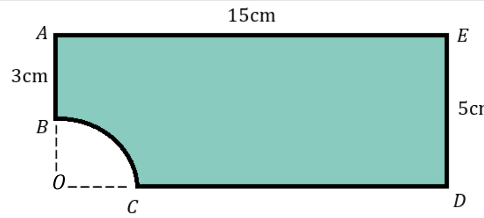


When the lamina is in limiting equilibrium, the inclination angle of the plane, α, is the same as the angle between the line drawn down from the centre of mass to the edge of the lamina's line of contact with the line perpendicular to the plane. When the angle of the plane surpasses this, the line drawn from the centre of mass will fall outside of the base, causing the lamina to topple. For the right-hand diagram, $\theta > \alpha$, and so this lamina will topple.



Example 2: A 15cm x 5cm rectangle has a quarter disc cut out of its bottom left corner. The resulting body is the uniform lamina $ABCDE$. The lamina is placed on an incline plane with sufficient friction to prevent sliding. The angle that the plane makes with the horizontal is 0.3 rad. Will the lamina topple about point C?

Draw out the lamina, including the given measurements. Add in an origin, O, where the bottom left-hand corner should be to allow for centres of mass to be expressed as coordinates.



Radius of the circle: $5 - 3 = 2$cm

Find the centre of mass of the rectangle using its symmetry, and the centre for the quarter disc using the formula for sectors of circles. The value calculated using this formula is the distance of the centre of mass from the centre of the sector, along its line of symmetry. Find the x and y coordinates using this length and the line of symmetry will bisect the angle of the sector, and so is $\frac{\pi}{4}$.

Centre of mass of the rectangle is $\binom{\bar{x}_r}{\bar{y}_r} = \binom{\frac{15}{2}}{\frac{5}{2}}$ by symmetry.

The centre of mass of a sector of a circle of radius r with angle 2α at the centre is $\frac{2r\sin(\alpha)}{3\alpha}$ along the line of symmetry of the sector, away from its centre. This circle has a radius of 2cm. The angle for a quarter circle is $\frac{\pi}{2}$ and so the centre of mass of the quarter circle is $\frac{2(2)\sin\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4}} = \frac{8\sqrt{2}}{3\pi}$ cm from O.

Find its x and y components to find its centre of mass, $\binom{\bar{x}_c}{\bar{y}_c}$:

$$\binom{\bar{x}_c}{\bar{y}_c} = \begin{pmatrix} \frac{8\sqrt{2}}{3\pi}\cos\left(\frac{\pi}{4}\right) \\ \frac{8\sqrt{2}}{3\pi}\sin\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{8}{3\pi} \\ \frac{8}{3\pi} \end{pmatrix}$$

Find the centre of mass of the body by subtracting the centre of mass of the quarter disc from that of the rectangle, each multiplied by their areas. As the laminae are uniform, their mass and area are proportional, and so their areas can be used within calculations. Find the angle between the line joining C to the centre of mass of the line perpendicular to the base of the lamina. Since the inclination angle is smaller than this, the lamina will not topple.

For the centre of mass of the whole body:

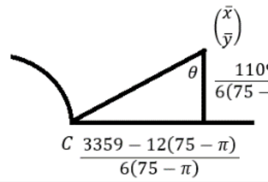
$$\left((15\cdot 5) - \left(\frac{\pi \cdot 2^2}{4}\right)\right)\binom{\bar{x}}{\bar{y}} = (15\cdot 5)\begin{pmatrix} \frac{15}{2} \\ \frac{5}{2} \end{pmatrix} - \left(\frac{\pi \cdot 2^2}{4}\right)\begin{pmatrix} \frac{8}{3\pi} \\ \frac{8}{3\pi} \end{pmatrix}$$

$$\therefore \binom{\bar{x}}{\bar{y}} = \frac{1}{6(75-\pi)}\binom{3359}{1109}$$

Find the angle between the line joining C, with coordinates $\binom{2}{0}$, to the centre of mass with the vertical:

$$\frac{1}{6(75-\pi)}\binom{3359}{1109} - \binom{2}{0} = \frac{1}{6(75-\pi)}\binom{3359 - 12(75-\pi)}{1109}$$

$$\theta = \tan^{-1}\left(\frac{3359 - 12(75-\pi)}{1109}\right) = 1.15 \text{ rad}$$



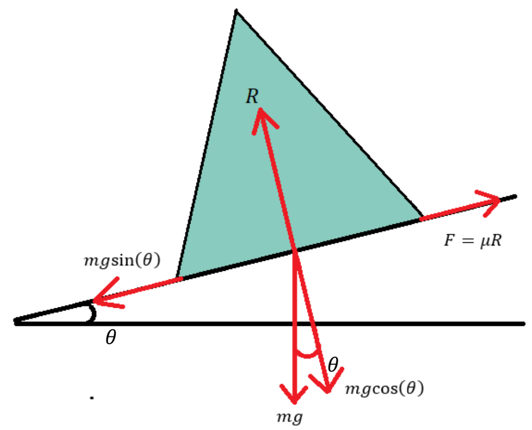
Since $0.3 < 1.15$, the lamina won't topple.

Conditions for a Lamina to Slide on an Inclined Plane

A lamina will slide down a slope if the component of its weight parallel to the slope is larger than the frictional force acting against its motion, and so the maximum friction provided by the surface of an inclined plane may not be sufficient to prevent sliding. In some cases, the angle required for the component of weight to be larger than the maximum friction is be smaller than the angle at which the lamina's centre of mass will lie outside its line of contact with the base. In this case, the lamina will slide before it topples. If the opposite is true, the lamina will topple first.

Example 4: A uniform solid cone of height 28cm and base radius 10cm rests on a rough plane. The angle that the plane makes with the horizontal, θ, begins to increase from 0. Given that the coefficient of friction, μ, between the cone and the surface is $\frac{3}{7}$, determine whether the cone will slide before it topples.

Draw the cone on top of an inclined plane and draw on the forces. The component of the weight of the cone acting down the slope is $mg\sin(\theta)$, and perpendicular to the plane is $mg\cos(\theta)$. The frictional force F is given by the product of the coefficient of friction and the normal reaction force, R.



Find the value of θ beyond which the cone will slide by forming an equation based on the resultant force being directed down the slope.

Since the resultant force perpendicular to the plane is zero:

$$mg\cos(\theta) = R \therefore F = \mu mg\cos(\theta)$$

For the cone to slide, the resultant force must be down the slope:

$$mg\sin(\theta) - \mu mg\cos(\theta) > 0$$

$$\therefore \sin(\theta) > \mu\cos(\theta) \Rightarrow \tan(\theta) > \mu$$

Given that $\mu = \frac{3}{7}$, the cone will slide when

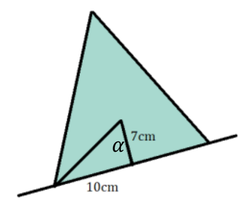
$$\theta > \tan^{-1}\left(\frac{3}{7}\right) = 0.405 \text{ rad}$$

Find the centre of mass of the solid cone using the known formula. Then, use this to find the angle required for limiting equilibrium (for toppling). When θ is beyond this angle, the cone will topple.

The centre of mass of the solid cone is

$$\frac{h}{4} = \frac{28}{4} = 7\text{cm}$$

vertically above the midpoint of the base (given in the formula booklet).



The angle α is given by

$$\tan(\alpha) = \frac{10}{7} \Rightarrow \alpha = 0.960 \text{ rad}$$

Base the conclusion on which outcome happens first, given the angle is increasing from 0.

Since

$$0.405 < 0.960,$$

a smaller inclination angle is required for the cone to slide down the slope than for it to topple. Therefore, it will slide before it topples.