# Centre of Mass and Moments III Cheat Sheet (A Level Only)



### Finding the Centre of Mass of a Lamina by Integration

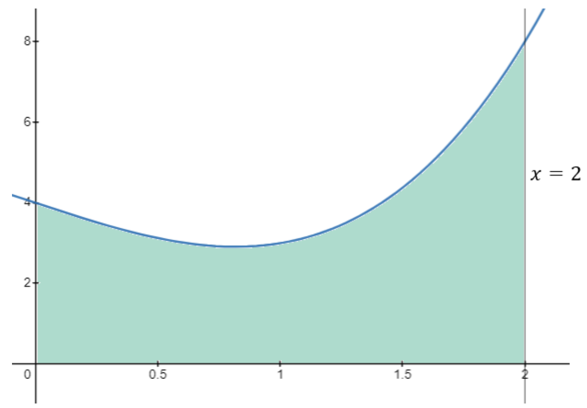
Laminas may be modelled as a function $f(x)$ which defines their shape in the $x$-$y$ plane. In these cases, integration can be used to find their centre of mass. The $x$ and $y$ coordinates of the lamina's centre of mass are calculated separately. For a uniform lamina defined by $f(x)$ between the lines $x = a, y = 0$, and $x = 0$, the coordinates $\binom{\bar{x}}{\bar{y}}$ of its centre of mass are given by the integrals

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx}, \qquad \bar{y} = \frac{1}{2}\frac{\int_0^a (f(x))^2 dx}{\int_0^a f(x)dx}.$$

Here, $\int_0^a f(x)dx$ is the area of the lamina. For a uniform lamina, mass is directly proportional to area, and so area can be used to find the centre of mass.

Example 1: A uniform lamina has three straight edges along the lines $x = 0, x = 2$ and $y = 0$. The fourth edge is modelled by the curve $f(x) = x^3 - 2x + 4$. Find the coordinates of the lamina's centre of mass.

This is the shape of the lamina in the $x$-$y$ plane, with the straight edge $x = 2$ indicated on the right.



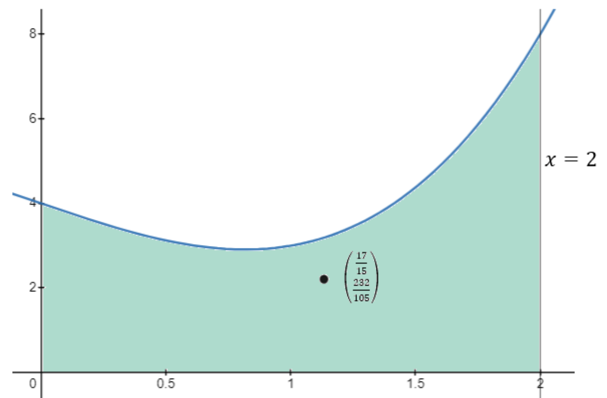

Find the $x$ and $y$ coordinates of the centre of mass using the formulae with $f(x) = x^3 - 2x + 4$, and 2 as the upper $x$ limit.

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx} = \frac{\int_0^2 x(x^3 - 2x + 4)dx}{\int_0^2 x^3 - 2x + 4\, dx} = \frac{\left[\frac{x^5}{5} - \frac{2x^3}{3} + 2x^2\right]_0^2}{\left[\frac{x^4}{4} - x^2 + 4x\right]_0^2} = \frac{\frac{136}{15}}{8} = \frac{17}{15}$$

$$\bar{y} = \frac{1}{2}\frac{\int_0^a (f(x))^2 dx}{\int_0^a f(x)dx} = \frac{\int_0^2 (x^3 - 2x + 4)^2 dx}{2\int_0^2 x^3 - 2x + 4dx} = \frac{\int_0^2 x^6 - 4x^4 + 8x^3 + 4x^2 - 16x + 16dx}{16} = \frac{\frac{3712}{105}}{16} = \frac{232}{105}$$

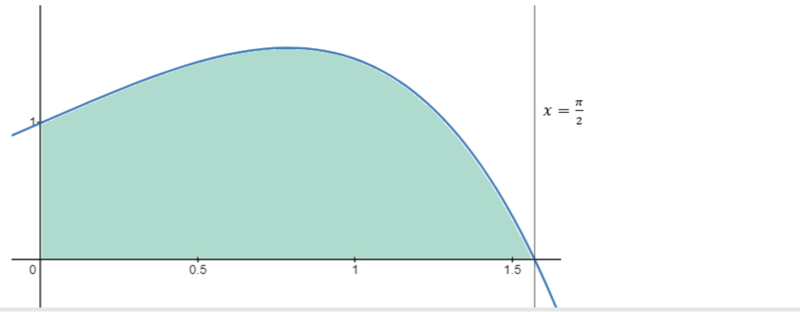
$$\therefore \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \frac{17}{15} \\ \frac{232}{105} \end{pmatrix}$$

The lamina with its centre of mass shown. It is to the right of the midpoint of its base, which is expected since the function takes larger values towards the right half of the lamina, and so there is a bigger contribution towards the total area from this side.




Example 2: A uniform lamina has three straight edges along lines $x = 0, x = \frac{\pi}{2}$ and $y = 0$. The fourth edge is modelled by the curve $f(x) = e^x \cos(x)$. Find the $x$ coordinate of the lamina's exact centre of mass in terms of $e$ and $\pi$.

This is the shape of the lamina in the $x$-$y$ plane, with the straight edge $x = \frac{\pi}{2}$ indicated on the right.




Begin by finding the area of the lamina. Use integration by parts, $\int u dv = uv - \int v du$, twice, arriving at an expression involving the original integral. Gather the multiples of the original integral on the left-hand side and divide through by a factor of two. This gives a value for the area of this lamina.

$$\int_0^{\frac{\pi}{2}} e^x \cos(x)\, dx$$

Let

$$u = \cos(x), dv = e^x \Rightarrow \frac{du}{dx} = -\sin(x), v = e^x$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \cos(x)\, dx = [\cos(x)\, e^x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \sin(x)\, e^x dx$$

Let

$$u = \sin(x), dv = e^x \Rightarrow \frac{du}{dx} = \cos(x), v = e^x$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \sin(x)\, dx = [\sin(x)\, e^x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(x)\, e^x$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x \cos(x)\, dx = [\cos(x)\, e^x]_0^{\frac{\pi}{2}} + [\sin(x)\, e^x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(x)\, e^x \Rightarrow \int_0^{\frac{\pi}{2}} e^x \cos(x)\, dx = \frac{1}{2}\left[e^x(\cos(x) + \sin(x))\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

Evaluate the integral in the numerator of the formula for $\bar{x}$. Begin by using integration by parts. In the previous step, the result $\int e^x \cos(x)\, dx = \frac{e^x}{2}(\cos(x) + \sin(x))$ was derived. Use this when setting up the integration by parts. Then, evaluate $\int_0^{\frac{\pi}{2}} e^x \sin(x) dx$ using the integral equations from the previous step. Combine these results to evaluate the integral in the numerator, and then divide by the result for the area of the lamina. This will give the $x$ coordinate of the centre of mass.

$$\bar{x} = \frac{\int_0^a xf(x)dx}{\int_0^a f(x)dx} = \frac{\int_0^{\frac{\pi}{2}} xe^x \cos(x)\, dx}{\int_0^{\frac{\pi}{2}} e^x \cos(x)\, dx} = \frac{\int_0^{\frac{\pi}{2}} xe^x \cos(x)\, dx}{\frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)}$$

Let

$$u = x, dv = e^x \cos(x) \Rightarrow \frac{du}{dx} = 1, v = \frac{1}{2}e^x(\cos(x) + \sin(x))$$

$$\therefore \int_0^{\frac{\pi}{2}} xe^x \cos(x)\, dx = \frac{1}{2}\left[xe^x(\cos(x) + \sin(x))\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} e^x \cos(x) + e^x \sin(x)\, dx$$

From the previous calculation,

$$\int_0^{\frac{\pi}{2}} e^x \sin(x)\, dx = [\sin(x)\, e^x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \cos(x)\, e^x = e^{\frac{\pi}{2}} - \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) = \frac{1}{2}\left(e^{\frac{\pi}{2}} + 1\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} xe^x \cos(x)\, dx = \frac{1}{2}\left[xe^x(\cos(x) + \sin(x))\right]_0^{\frac{\pi}{2}} - \frac{1}{4}\left(\left(e^{\frac{\pi}{2}} + 1\right) + \left(e^{\frac{\pi}{2}} - 1\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}e^{\frac{\pi}{2}}\right) - \frac{1}{2}e^{\frac{\pi}{2}} = \frac{e^{\frac{\pi}{2}}}{4}(\pi - 2)$$

$$\therefore \bar{x} = \frac{\frac{e^{\frac{\pi}{2}}}{4}(\pi - 2)}{\frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)} = \frac{e^{\frac{\pi}{2}}(\pi - 2)}{2\left(e^{\frac{\pi}{2}} - 1\right)}$$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