Centres of Mass and Moments I Cheat Sheet (A Level Only)

The Centre of Mass of a System of Particles

Particles in a Straight Line

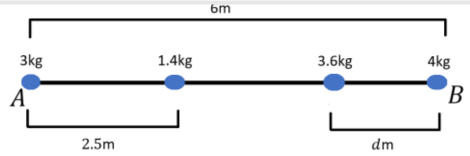
Consider particles lying in a straight line. If there are two particles present, and their masses are equal, their centre of mass will lie at the midpoint of the line connecting them. If one particle were to be heavier, the centre of mass would be closer to the larger mass, since it contributes more to the total mass. If there are n masses in a system, each of mass m_i, on a straight line, these can be modelled as a single particle of mass M, where $M = \sum_{i=1}^{n} m_i$, whose mass is distrubuted across the line. In this case, the position of the centre of mass of the system, and thus of the mass M, is given by the following weighted average:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{M},$$

where x_i is the position on the line of mass m_i. Note that these positions will be relative to some end of the line connecting the particles.

Example 1: A light rod AB of length 6m has four masses attached to it. A mass of 3kg is attached at A, a mass of 4kg is attached at B, a mass of 1.4kg is attached 2.5m from A, and a mass of 3.6kg is attached dm from B. Is it possible for the centre of mass to be 2m from A? If so, find d.

Begin by drawing a diagram of the situation. This will help to visualise what is going on, and what needs to be found.



Find the total mass of the system by summing the individual masses and set up an equation for the centre of mass using the weighted average formula, noting that every distance needs to be relative to A. Base the conclusion of the problem on where it would mean the 3.6kg mass would have to sit, considering the length of the beam.

$$M = 3 + 1.4 + 3.6 + 4 = 12\text{kg}$$

dm from B is $(6-d)$m from A.

$$2 = \frac{1}{12}(3(0) + 4(6) + 1.4(2.5) + 3.6(6-d)) = \frac{1}{12}(49.1 - 3.6d)$$

$$\therefore 24 = 49.1 - 3.6d \Rightarrow 3.6d = 49.1 - 24 = 25.1$$

$$\therefore d = \frac{25.1}{3.6} = 6.97\text{m}$$

∴ It is not possible for the centre of mass of this system to be 2m from A since this would require the 3.6kg mass to be attached beyond the end of the rod.

Centre of Mass of Points in a Plane

Particles can also be arranged within a plane. In this case, the centre of mass is found separately for the x and y directions, in the same way as above. For n masses, each of mass m_i, where $M = \sum_{i=1}^{n} m_i$, and each with position vector $\binom{x_i}{y_i}$ relative to some fixed origin, the centre of mass of the system has position vector

$$\binom{\bar{x}}{\bar{y}} = \frac{1}{M}\left(m_1\binom{x_1}{y_1} + m_2\binom{x_2}{y_2} + \cdots + m_n\binom{x_n}{y_n}\right).$$

Example 2: Three particles of masses 2kg, 4kg and 6kg are positioned in a plane at coordinates $\binom{-3}{-3}$, $\binom{-3}{9}$ and $\binom{1}{7}$ respectively. Find the distance of the centre of mass of this system from the position of the 4kg mass.

Begin by finding the total mass of the system, and then multiply the position vectors with the corresponding masses. Sum these vectors and divide by the total mass to find the centre of mass of the system.

$$M = 2 + 4 + 6 = 12\text{kg}$$

$$\binom{\bar{x}}{\bar{y}} = \frac{1}{12}\left(2\binom{-3}{-3} + 4\binom{-3}{9} + 6\binom{1}{7}\right) = \frac{1}{12}\binom{-6-12+6}{-6+36+42} = \binom{-1}{6}$$

Find the distance of the centre of mass from the position vector of the 4kg mass by taking their position vectors away from each other, and then calculating the magnitude of this vector by taking the square root of the sum of the squares of its components.

Let $A = \binom{-1}{6}$, $B = \binom{-3}{9}$

Then,

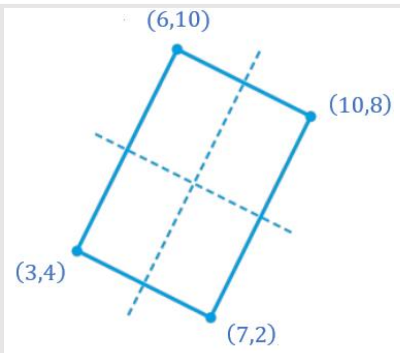
$$|\overrightarrow{AB}| = |B - A| = \left|\binom{-3+1}{9-6}\right| = \left|\binom{-2}{3}\right| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

Centres of Mass of Uniform Laminas by Symmetry

Laminas are two-dimensional objects which lie in a plane and are assumed to have zero thickness. A uniform lamina has constant mass per unit area. Laminas can take various shapes, and their centres of mass lie on their axis of symmetry. If a lamina has more than one axis of symmetry, its centre of mass will lie on the intersection of these axes. Rods, inflexible bodies with zero cross sectional area, also have their centre of mass along their axis of symmetry, this being along the midpoint of the rod. A quick way to find the centre of a symmetrical object, and therefore its centre of mass, is to find the mean of its co-ordinates.

Example 3: Find the centre of mass of a uniform rectangular lamina with vertices at $(3,4)$, $(7,2)$, $(10,8)$ and $(6,10)$.

Draw the shape along with its axes of symmetry. The centre of mass of this lamina will be at its centre, as this is the intersection point of the two axes of symmetry. The centre of this lamina will be at the mean of the coordinates, i.e., the mean of all the x and y values of the co-ordinates. Find these averages to find the centre of mass.



$$\bar{x} = \frac{1}{4}(3 + 6 + 7 + 10) = \frac{13}{2}$$

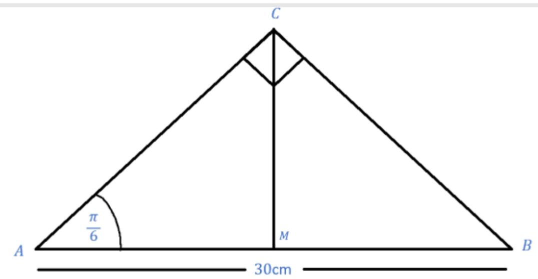
$$\bar{y} = \frac{1}{4}(4 + 10 + 2 + 8) = 6$$

∴ The centre of mass is $\binom{\bar{x}}{\bar{y}} = \binom{\frac{13}{2}}{6}$

A similar process is done to find the centre of mass of an equilateral triangle, whereby the centre of mass is at the intersection of the medians (lines drawn from each vertex to the midpoint of the opposite side). When a triangle is not equilateral, the centre of mass is still at the intersection of the medians. This point is found by taking the average of the three vertices of a triangle and is found to be $\frac{2}{3}$ along any median from any vertex of the triangle. This is given in the formula booklet.

Example 4: Find the centre of mass of the following triangular lamina from the vertex C.

The centre of mass is $\frac{2}{3}$ along the median from any vertex, so it is $\frac{2}{3}$ along the median from C to the midpoint M of AB. Find the length of this median using trigonometry with the angle A of $\frac{\pi}{6}$ radians, and then multiply this length by $\frac{2}{3}$.



Length of CM:

$$\frac{30}{2}\tan\left(\frac{\pi}{6}\right) = 15 \cdot \frac{\sqrt{3}}{3}$$

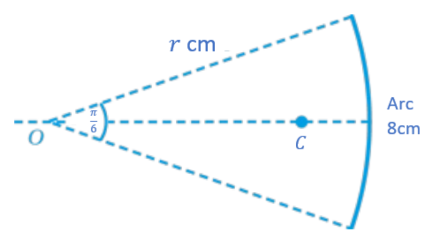
∴ distance of the centre of mass from C is

$$\frac{2}{3} \cdot \frac{15\sqrt{3}}{3} = \frac{10\sqrt{3}}{3}\text{cm}$$

There is also a formula for the centre of mass of a lamina in the shape of a sector of a circle. For the sector of a circle of radius r, and of angle size 2α radians, the centre of mass of the lamina is $\frac{2r\sin(\alpha)}{3\alpha}$ from the centre of a circle, along the axis of symmetry. There is a similar formula for an arc of a wire bent to form the arc of a sector, whereby the centre of mass is $\frac{r\sin(\alpha)}{\alpha}$ from the centre of the circle. Both formulas are provided in the formula book.

Example 5: An 8cm piece of uniform wire is bent to form an arc of a sector of angle $\frac{\pi}{6}$ radians. Find the distance of the centre of mass, C, of the wire from the centre of the circle, O.

Begin by finding the radius of the circle, using arc length $= r\theta$ where θ is the angle of the sector. Then, use the given formula to find the distance of the centre from O, remembering to half the angle.



$$8 = r \cdot \frac{\pi}{6} \Rightarrow r = \frac{48}{\pi}\text{cm}$$

∴ the distance OC is

$$\frac{r\sin(\alpha)}{\alpha} = \frac{\left(\frac{48}{\pi}\right)\sin\left(\frac{\pi}{6} \cdot \frac{1}{2}\right)}{\frac{\pi}{6} \cdot \frac{1}{2}} = 15.1\text{cm}$$