OCR A Further Maths A-level

Pure Core

Formula Sheet

Provided in formula book | Not provided in formula book

Complex Numbers

The Language of Complex Numbers

Cartesian form of a complex number	$z = a + ib$ $a = Re(z), \quad b = Im(z)$
Modulus-argument form of a complex number	$z = a + bi, \quad \lvert z \rvert = r = \sqrt{a^2 + b^2},$ $\arg(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right)$ $z = r(\cos\theta + i\sin\theta) = [r, \theta]$
Exponential form	$z = a + bi = re^{i\theta}$ $r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$
Complex conjugate of a complex number	$z = a + ib$ has complex conjugate $z^* = a - bi$ $\left(re^{i\theta}\right)^* = re^{-i\theta}$

Basic Operations

Multiplication in modulus-argument form	$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert, \ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
Division in modulus-argument form	$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Loci

Loci of points z such that $\lvert z-a\rvert = k$	Circle of radius k centred on $(Re(a), Im(a))$
Loci of points z such that $\lvert z-a\rvert = \lvert z-b\rvert$	Perpendicular bisector of the line from a to b
Loci of points z such that $\arg(z-a) = \alpha$	Half-line starting from a making an angle α with the real axis

De Moivre's Theorem

De Moivre's theorem	$z^n = \left(r\left(\cos(\theta) + i\sin(\theta)\right)\right)^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$
$\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$	Let $z = e^{i\theta}$. Then, $\cos(n\theta) = \frac{z^n + z^{-n}}{2}$, $\sin(n\theta) = \frac{z^n - z^{-n}}{2i}$

n^{th} Roots of a Complex Number

Solving to find the n^{th} roots of a complex number w	$z^n = w$. Let $z = r_1e^{i\theta_1}, w = r_2e^{i\theta_2}$ $r^ne^{in\theta_1} = r_2e^{i\theta_2}$ $re^{i\theta_1} = \sqrt[n]{r_2}e^{i\left(\frac{\theta_2}{n} + \frac{2k\pi}{n}\right)}$ $r = \sqrt[n]{r_2}$, $\theta_k = \dfrac{\theta_2}{n} + \dfrac{2k\pi}{n}, k \in [0, n-1]$
Geometry of the n^{th} roots of a complex number	The n roots of $z^n = w$ will form a regular polygon in the complex plane, with vertices on a circle centred at the origin

n^{th} Roots of Unity

An n^{th} root of unity	A complex number z is an n^{th} root of unity if $z^n = 1$. They are $\left\{1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \dots, e^{\frac{2(n-1)\pi i}{n}}\right\} = \{1, \omega_1, \omega_2, \dots, \omega_{n-1}\}$
Sum of the roots of unity	$1 + \omega_1 + \cdots + \omega_{n-1} = 0$

Matrices

The Language of Matrices

An $m \times n$ matrix has m rows and n columns	$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
The null matrix has zeros in every entry	$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$
The identity matrix, I, is a square matrix with 1s on the leading diagonal and 0s elsewhere	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$
The transpose of a matrix $\boldsymbol{A}$, $\boldsymbol{A^T}$, swaps the rows and columns of $\boldsymbol{A}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

Addition and Multiplication

Addition and subtraction are performed element wise	$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \pm \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} =$ $\begin{pmatrix} a_1 \pm a_2 & b_1 \pm b_2 & c_1 \pm c_2 \\ d_1 \pm d_2 & e_1 \pm e_2 & f_1 \pm f_2 \\ g_1 \pm g_2 & h_1 \pm h_2 & i_1 \pm i_2 \end{pmatrix}$
Matrix multiplication by a scalar	$k\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ak & bk & ck \\ dk & ek & fk \\ gk & hk & ik \end{pmatrix}$
Matrix multiplication	$A: m \times n$ matrix, $B: n \times p$ matrix $(\boldsymbol{AB})_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ $\boldsymbol{AB}: n \times p$ matrix
Associativity and non-commutativity of matrix multiplication	$\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ $\boldsymbol{AB} \neq \boldsymbol{BA}$ (In general. If this is true, $\boldsymbol{A}$ and $\boldsymbol{B}$ **commute**

2D Linear Transformations

3D Rotations

Transformation	Associated Matrix
Reflection in x axis.	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in y axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Enlargement by scale factor a	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
Stretch parallel to x axis by scale factor a	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$
Stretch parallel to y axis by scale factor a	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$
Reflection in line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Anticlockwise rotation by an angle θ	$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
Transformation with matrix $\boldsymbol{A}$ followed by transformation with matrix **B**	$\boldsymbol{BA}$

The direction of positive rotation is taken to be anticlockwise when looking towards the origin from the positive side of the axis of rotation.

Rotation around x axis by an angle θ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$
Rotation around y axis by an angle θ	$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$
Rotation around z axis by an angle θ	$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Invariance Under Transformations

Invariant point $\begin{pmatrix} x \\ y \end{pmatrix}$ under a transformation $\boldsymbol{M}$	$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
Invariant line l	The image of any point on l is also on l

Determinants

Determinant of a 2 × 2 matrix	$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
Determinant of a matrix product	$\det \boldsymbol{AB} = \det \boldsymbol{A} \times \det \boldsymbol{B}$
Determinant of a multiple of an $n \times n$ matrix	$\det(k\boldsymbol{A}) = k^n \det(\boldsymbol{A})$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Inverses of Matrices

Inverse matrix	$\boldsymbol{A}^{-1}$ is the inverse matrix of $\boldsymbol{A}$, such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
Singular matrix	$\det(\boldsymbol{A}) = 0 \Rightarrow \boldsymbol{A}^{-1}$ does not exist. $\boldsymbol{A}$ is *singular*
Inverse of a 2×2 matrix	$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $ad - bc \neq 0$
Cofactor of an element – determinant of the matrix without the element's row and column	Cofactor of element a in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$
Cofactor matrix of $\boldsymbol{A}$ – made of the cofactors of all elements of $\boldsymbol{A}$	Denoted by $\boldsymbol{C}$
Inverse of a 3×3 matrix	$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{C}^T$
Inverse of a matrix product	$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
Inverse of a transformation	For a transformation given by matrix $\boldsymbol{M}$, its inverse is given by $\boldsymbol{M}^{-1}$

Solutions of Simultaneous Equations

Condition for a system of equations $\boldsymbol{Mr} = \boldsymbol{a}$ to have a unique solution	$\det(\boldsymbol{M}) \neq 0$
For systems with no unique solution	Eliminate a variable from the system. If this leads to consistent equations, there are infinitely many solutions. If the equations are inconsistent, there are no solutions.

Intersections of Planes – the Geometry of the Systems of Equations

A system of three linear equations in three variables will define three planes in 3D space. The geometry of these planes relates to how many solutions the system of equations has.

There is a unique solution to the system	The planes defined by the equations intersect in one point.
There are infinitely many solutions to the system	The planes meet along a line, and form a sheaf.
There are no solutions to the system	Either all planes are parallel, two planes are parallel, or the planes form a triangular prism.

Further Vectors

Vector and Cartesian Forms of an Equation of a Straight Line

Vector equation of a line through the point $\boldsymbol{a}$ parallel to the vector $\boldsymbol{b}$	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
Cartesian equation of a line in 3D	For $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \lambda \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$, writing λ in terms of x, y and z: $\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$

Vector, Cartesian, and Point-Normal Forms of a Plane in 3D

Vector equation of a plane containing the point with position vector $\boldsymbol{a}$, and containgin vectors $\boldsymbol{b}$ and $\boldsymbol{c}$	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
Point-normal equation of a plane. $\boldsymbol{a}$ is the position vector of a point in the plane, and $\boldsymbol{n}$ is the normal to the plane	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
Cartesian equation of a plane in 3D. Here, $\boldsymbol{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ is the normal vector to the plane	$n_1x + n_2y + n_3z = d$

Scalar Product

Scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$	$\boldsymbol{a}\cdot\boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(\theta)$
Angle θ between two vectors $\boldsymbol{a}$, $\boldsymbol{b}$, or between two lines with these direction vectors	$\theta = cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}\right)$
Condition for $\boldsymbol{a}$ and $\boldsymbol{b}$ to be perpendicular vectors	$\boldsymbol{a}\cdot\boldsymbol{b} = 0$
Angle θ between two planes is the same as the angle between their normal vectors	$\pi_1 : \boldsymbol{r}\cdot\boldsymbol{n_1} = \boldsymbol{a_1}\cdot\boldsymbol{n_1}, \quad \pi_2 : \boldsymbol{r}.\boldsymbol{n_2} = \boldsymbol{a_2}\cdot\boldsymbol{n_2},$ $\theta = cos^{-1}\left(\frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{\lvert\boldsymbol{n_1}\rvert\lvert\boldsymbol{n_2}\rvert}\right)$
Angle θ between a line and a plane	$\boldsymbol{r} = \boldsymbol{a_1} + \lambda\boldsymbol{d}, \quad \pi = \boldsymbol{r}\cdot\boldsymbol{n} = \boldsymbol{a_2}\cdot\boldsymbol{n}$ $\sin(\theta) = \left(\frac{\boldsymbol{n}\cdot\boldsymbol{d}}{\lvert\boldsymbol{n}\rvert\lvert\boldsymbol{d}\rvert}\right)$

Intersections

Intersection type	$\boldsymbol{r_1} = \boldsymbol{a_1} + \lambda_1\boldsymbol{b_1}, \quad \boldsymbol{r_2} = \boldsymbol{a_2} + \lambda_2\boldsymbol{b_2}$
Parallel lines	$\boldsymbol{b_1} = \mu\boldsymbol{b_2}$
Intersecting lines	There exist values of λ_1 and λ_2 such that $\boldsymbol{r_1} = \boldsymbol{r_2}$
Skew	No such λ_1 and λ_2 as above exist
Intersection of a line and a plane	$\boldsymbol{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \boldsymbol{\pi = cx + dy + fz = g}$ If there exists a λ such that $c(a_1 + \lambda b_1) + d(a_2 + \lambda b_2) + f(a_3 + \lambda b_3) = g$ Then the line and plane intersect. If no such λ exists, they do not intersect.

Vector Product

Vector Product – gives a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$	$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - b_3a_1 \\ a_1b_2 - b_1a_2 \end{pmatrix}$

Shortest Distances

Shortest distance, D, between two parallel lines	$r_1 = \boldsymbol{a} + \lambda_1\boldsymbol{d}, \quad \boldsymbol{r_2} = \boldsymbol{b} + \lambda_2\boldsymbol{d}$ $D = \lvert\boldsymbol{a} - \boldsymbol{b}\rvert \sin(\theta)$ where $\cos(\theta) = \frac{(\boldsymbol{a} - \boldsymbol{b}) \cdot \boldsymbol{d}}{\lvert\boldsymbol{a} - \boldsymbol{b}\rvert\lvert\boldsymbol{d}\rvert}$
Shortest distance, D, between a point and a line	For a point with co-ordinates (x_1, y_1), and a line given by $ax + by = c$: $D = \frac{\lvert ax_1 + by_1 - c\rvert}{\sqrt{a^2 + b^2}}$
Shortest distance, D, between a point and a plane	For a point with position vector $\boldsymbol{b}$ and a plane with equation $\boldsymbol{r} \cdot \boldsymbol{n} = p$: $D = \frac{\lvert\boldsymbol{b} \cdot \boldsymbol{n} - p\rvert}{\lvert\boldsymbol{n}\rvert}$
Shortest distance, D, between two skew lines	For points $\boldsymbol{a}$, $\boldsymbol{b}$ on the lines and a mutually perpendicular vector $\boldsymbol{n}$: $D = \frac{(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{n}}{\lvert\boldsymbol{n}\rvert}$

Further Algebra

Roots of Equations

Relationship between the roots and coefficients of a quadratic polynomial	Let p and q be roots of $ax^2 + bx + c = 0$. Then, $$p + q = -\frac{b}{a}, \qquad pq = \frac{c}{a}$$
Relationship between the roots and coefficients of a cubic polynomial	Let p, q, and r be the roots of $ax^3 + bx^2 + cx + d = 0$. Then, $$p + q + r = -\frac{b}{a}, \qquad pq + qr + rp = \frac{c}{a}, \qquad pqr = -\frac{d}{a}$$
Relationship between the roots and coefficients of a quartic polynomial	Let p, q, r and s be the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$. Then, $$p + q + r + s = -\frac{b}{a}, \qquad pq + pr + ps + qr + qs + rs = \frac{c}{a},$$ $$pqr + pqs + prs + qrs = -\frac{d}{a}, \qquad pqrs = \frac{e}{a}$$

Transformations of Equations

Transformation of the roots of an equation, given a transformation of the equation	Let an equation in x have root $x = p$. Given a substitution $u = f(x)$, the transformed equation has a root $u = f(p)$

Partial Fractions

The partial fraction decomposition for denominators of the form $\frac{1}{q^2+x^2}$	$$\frac{f(x)}{(x-p)(x^2+q^2)} = \frac{A}{x-p} + \frac{Bx+c}{x^2+q^2}$$

Series

Summation of Series

$\sum_{r=1}^{r=n} k = nk$	$\sum_{r=1}^{r=n} r = \frac{n}{2}(n+1)$
$\sum_{r=1}^{r=n} r^2 = \frac{1}{6}n(n+1)(2n+1)$	$\sum_{r=1}^{r=n} r^3 = \frac{1}{4}n^2(n+1)^2$
$\sum(u_r + v_r) = \sum u_r + \sum v_r$	$\sum c u_r = c \sum u_r$

The Method of Differences

If a general term of a series, v_r, can be written in the form $v_r = f(r+1) - f(r)$ for some function f, then

$$\sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

Hyperbolic Functions

Definitions, Domains, Derivatives, and Integrals

Function	Definition and Domain	Derivative	Indefinite Integral
$\sinh(x)$	$\frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$	$\frac{d(\sinh x)}{dx} = \cosh x$	$\cosh(x) + c$
$\cosh(x)$	$\frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$	$\frac{d(\sinh x)}{dx} = \cosh x$	$\sinh(x) + c$
$\tanh(x)$	$\frac{e^x - e^{-x}}{e^x + e^{-x}}, x \in \mathbb{R}$	$\frac{d(\tanh x)}{dx} = \frac{1}{\cosh^2 x}$	$\ln\lvert cosh(x)\rvert + c$
$\cosh^2(x) - \sinh^2(x) \equiv 1$			

Inverse Hyperbolic Functions

Function	Domain	Logarithmic Form
$\sinh^{-1}(x)/\operatorname{arsinh}(x)$	$x \in \mathbb{R}$	$\ln(x + \sqrt{x^2 + 1})$
$\cosh^{-1}(x)/\operatorname{arcosh}(x)$	$x \geq 1$	$\ln(x + \sqrt{x^2 - 1})$
$\tanh^{-1}(x)/\operatorname{artanh}(x)$	$-1 < x < 1$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

Further Calculus

Maclaurin Series

Maclaurin Series Expansion of a function $f(x)$
$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^r}{r!}f^r(0) + \cdots$

Standard Maclaurin Series

e^x	$1 + x + \frac{x^2}{2!} + \cdots \frac{x^r}{r!} + \cdots$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{r+1}\frac{x^r}{r} + \cdots$
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots$
$(1+x)^n$	$1 + nx + \frac{n(n-1)}{2!}x + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$

Improper Integrals

The integrand is undefined at a one of the limits of integration	Upper limit: $\int_a^k f(x)dx = \lim_{b\to k} \int_a^b f(x)dx$ Lower limit: $\int_k^c f(x)dx = \lim_{b\to k} \int_b^c f(x)dx$
The integrand is undefined at a point $x = k$ within the domain of integration	$\int_a^c f(x)dx = \lim_{b\to k} \int_a^b f(x)dx + \lim_{b\to k} \int_b^c f(x)dx$
The limit(s) of integration extend to infinity	$\int_a^\infty f(x)dx = \lim_{b\to\infty} \{I(b) - I(a)\}$ Where $I(k)$ is the integral evaluated at the point k
If these integrals have finite limits, they **converge**. Else, they **diverge**	

Volumes of Solids of Revolution

Revolving $y = f(x)$ between $x = a$ and $x = b$ 2π rad around the x-axis	$V = \pi \int_a^b y^2 dx$
Revolving $y = f(x)$ between $y = c$ and $y = d$ 2π rad around the y-axis	$V = \pi \int_c^d x^2 dy$
Volume of revolution of the region between the two curves $g(x)$ and $f(x)$, where $g(x) > f(x)$, and $g(a) = f(a), g(b) = f(b)$	$V = \pi \int_a^b (g(x)^2 - f(x)^2)dx$
Volume of revolution generated by rotating the curve with parametric equations $x = f(t), y = g(t)$ between two points with parameter values of t_1 and t_2	Rotation about the x-axis: $V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt} dt$ Rotation about the y- axis: $V = \pi \int_{t_1}^{t_2} x^2 \frac{dx}{dt} dt$

Mean value of a function

Mean value of $f(x)$ on the interval $[a, b]$:

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Integration Using Partial Fractions

Expression Type	Partial Fraction Decomposition
$\frac{px+q}{(x+a)(x+b)}$	$\frac{A}{x+a}+\frac{B}{x+b}$
$\frac{px+q}{(x+a)^2}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}$
$\frac{px^2+qx+r}{(x+a)(x+b)(x+c)}$	$\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c}$
$\frac{px^2+qx+r}{(x+a)^2(x+b)}$	$\frac{A}{x+a}+\frac{B}{(x+a)^2}+\frac{C}{x+b}$
$\frac{px^2+qx+r}{(x+a)(x^2+b^2)}$	$\frac{A}{x+a}+\frac{Bx+C}{x^2+b^2}$

Differentiation of Inverse Trigonometric and Hyperbolic Functions

$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$	$\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{1+x^2}}$
$\frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2-1}}$
$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$	$\frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1-x^2}$

Integration of Four Specific Forms

$\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}\frac{x}{a} + c$	$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}\frac{x}{a} + c$
$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$	$\int \frac{1}{a^2+x^2} dx = \tan^{-1}\frac{x}{a} + c$

Polar Coordinates

Converting Between Polar and Cartesian Coordinates

Converting from cartesian coordinates $(x, y)$to polar coordinates (r, θ)	$r = x^2 + y^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$
Converting from polar to cartesian coordinates	$x = r\cos\theta,\ y = r\sin\theta$

Curve Sketching

$r = a$	Circle of radius a
$\theta = \alpha$	Half-line at an angle α to the x axis
$r = a\theta$	Spiral pattern
$\sin(n\theta)$ or $\cos(n\theta)$	Flower petal pattern with $2n$ petals if n is odd and n petals if n is even.
$r = a(b + \cos\theta)$	$\|b\| > 2$ gives an egg-shaped curve $1 < \|b\| < 2$ gives a dimpled egg $b = 1$ gives a cardioid

Area Enclosed by a Polar Curve $r(\theta)$

$$A = \frac{1}{2}\int r^2 d\theta$$

Differential Equations

Integrating Factor Method for First Order Differential Equations

To solve differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$, multiply through by the integrating factor.	
Integrating factor: $I(x) = e^{\int p(x)dx}$	**General solution:** $y(x) = \frac{1}{I(x)} \int I(x)Q(x)dx$

Second Order Homogenous Equations – General Solutions

Let the auxiliary equation have roots α and γ:

α and γ are real	$y = A\mathrm{e}^{\alpha x} + B\mathrm{e}^{\gamma x}$
$\alpha = \gamma$	$y = A\mathrm{e}^{-\alpha x} + Bx\mathrm{e}^{-\alpha x}$
α and γ are complex with $\alpha = a + bi, \gamma = a - bi$	$y = e^{ax}(A\cos bx + B\sin bx)$

Particular Solutions for Second Order Non-Homogenous Equations

For an equation of the form: $y'' + ay' + by = f(x)$

$f(x) = Ae^{cx}$	ke^{cx}
$f(x) = Ax^n + \cdots B$	$k_1x^n + k_2x^{n-1} + \cdots + k_{n+1}$
$f(x) = A\sin(cx) + B\cos(cx)$	$y = k_1\cos(cx) + k_2\sin(cx)$

Simple Harmonic Motion Equation

$\frac{d^2x}{xt^2} = -\omega^2 x$	General solution: $x = A\sin(\omega t) + B\cos(\omega t)$ $= R\sin(\omega t + \varphi)$
Time period, T, or a particle moving with simple harmonic motion	$T = \frac{2\pi}{\omega}$
Relationship between velocity and displacement for a particle moving with simple harmonic motion. Here, x is the displacement, and a is the maximum displacement	$v^2 = \omega^2(a^2 - x^2)$

Damped Simple Harmonic Motion

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$$

Type of damping	**Coefficient conditions**	**General solution**
Overdamping	$k^2 - 4\omega^2 > 0$	$x = Ae^{\alpha t} + Be^{\beta t}$
Critical damping	$k^2 - 4\omega^2 = 0$	$x = (A + Bt)e^{-\frac{k}{2}t}$
Underdamping	$k^2 - 4\omega^2 < 0$	$x = e^{-\frac{k}{2}t}\left(A\sin(qt) + B\cos(qt)\right)$