Edexcel Further Maths A-level

Core Pure

Formula Sheet

Provided in formula book

Not provided in formula book

Complex Numbers

Euler's relation	$e^{i\theta} = \cos\theta + i\sin\theta$

Rectangular coordinate form	$z = x + i\,y$
Exponential form	$z = re^{i\theta}$ $r = \lvert z\rvert = \sqrt{x^2 + y^2}$ $\theta = \arg(z) = \tan^{-1}\frac{y}{x}$

$$z = r(\cos\theta + i\sin\theta) = r(\cos(\theta + 2k\pi) + \; i\sin(\theta + 2k\pi))\ ,\ k \in \mathbb{Z}$$

De Moivre's Theorem

$$\{r(\cos\theta \ + \ \mathrm{i}\sin\theta)\}^n = r^n(\cos n\theta \ + \ \mathrm{i}\sin n\theta)$$

n[th] Roots of Unity

The solutions of $z^n = 1$ are given by $z = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} = e^{\frac{2\pi k}{n}i}$ for $n = 0, 1, 2, \ldots, n-1$.

Operations and Identities

For any two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$:	$z_1 \cdot z_2 = (r_1 \cdot r_2)e^{i(\theta_1+\theta_2)}$
	$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

$z + \frac{1}{z} = \ 2\cos\theta$
$z - \frac{1}{z} = 2i\sin\theta$
$z^n + \frac{1}{z^n} = 2\cos n\theta$
$z^n - \frac{1}{z^n} = 2i\sin n\theta$

Matrices

Elementary Operations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$$

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Matrix Multiplication

Matrices can be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second matrix.

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

Determinant

For a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:	$\det M = ad - bc$
For a 3×3 matrix $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$:	$\det(M) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Singular matrix	$\det M = 0$
Non-singular matrix	$\det M \neq 0$

Inverse Matrices

$$M^{-1}M = MM^{-1} = I$$

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $M^{-1} = \frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

For non-singular matrices A and B:	$(AB)^{-1} = B^{-1}A^{-1}$

Matrix Transformations

Anticlockwise rotation through θ about O:	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
Reflection in the line $y = (\tan\theta)x$:	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Summations and Series

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Maclaurin's and Taylor's Series

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^r}{r!}f^{(r)}(0) + \cdots$$

$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots$ for all x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r+1}\frac{x^r}{r} + \cdots \quad (-1 < x \leq 1)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$ for all x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots$ for all x

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^r \frac{x^{2r+1}}{2r+1} + \cdots \quad (-1 \leq x \leq 1)$

Calculus

Differentiation

$f(x)$	$f'(x)$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\text{sech}^2\, x$
$\text{arsinh}\, x$	$\frac{1}{\sqrt{1+x^2}}$
$\text{arcosh}\, x$	$\frac{1}{\sqrt{x^2-1}}$
$\text{artanh}\, x$	$\frac{1}{1-x^2}$
$\coth x$	$-\text{coshec}^2 x$
$\text{cosech}\, x$	$-\coth x\,\text{cosech}\, x$
$\text{sech}\, x$	$-\tanh x\,\text{sech}\, x$

Integration ($+$ *constant*, $a > 0$ **where appropriate)**

$f(x)$	$\int f(x)\ dx$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ln \cosh x$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right)$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\text{arcosh}\left(\frac{x}{a}\right), \quad \ln\{x + \sqrt{x^2 - a^2}\} \quad (x > a)$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\text{arsinh}\left(\frac{x}{a}\right), \quad \ln\{x + \sqrt{x^2 + a^2}\}$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a + x}{a - x}\right\rvert = \frac{1}{a}\text{artanh}\left(\frac{x}{a}\right) \quad (\lvert x\rvert < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x - a}{x + a}\right\rvert$

Surface Area of Revolution

Cartesian coordinates	$s_x = 2\pi \int y \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$
Parametric form	$s_x = 2\pi \int y \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)}\, dt$
Polar form	$s_x = 2\pi \int r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Arc Length

Cartesian coordinates	$s_x = \int \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$
Parametric form	$s_x = \int \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)}\, dt$
Polar form	$s_x = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

Vectors

Vector product	$\boldsymbol{a}\times\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta\,\hat{\boldsymbol{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$
Scalar product	$\boldsymbol{a}.\boldsymbol{b} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$

$$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b})$$

Cartesian and Vector Equations

If A is the point with position vector $\boldsymbol{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and the direction vector $\boldsymbol{b}$ is given by $\boldsymbol{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then the straight line through A with direction vector $\boldsymbol{b}$ has cartesian equation

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3} \; (=\lambda)$$

The plane through A with normal vector $\boldsymbol{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ has cartesian equation $n_1x + n_2y + n_3z + d = 0$ where $d = -\boldsymbol{a}.\boldsymbol{n}$

The plane through non-collinear points A, B and C has vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b}-\boldsymbol{a}) + \mu(\boldsymbol{c}-\boldsymbol{a}) = (1-\lambda-\mu)\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

The plane through the point with position vector $\boldsymbol{a}$ and parallel to $\boldsymbol{b}$ and $\boldsymbol{c}$ has equation $\boldsymbol{r} = \boldsymbol{a} + s\boldsymbol{b} + t\boldsymbol{c}$

The plane through non-collinear points A, B and C has vector equation $\boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{b}-\boldsymbol{a}) + \mu(\boldsymbol{c}-\boldsymbol{a}) = (1-\lambda-\mu)\boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$

Calculating Distances

The perpendicular distance of (α, β, γ) from $n_1x + n_2y + n_3z + d = 0$	$\frac{\lvert n_1\alpha + n_2\beta + n_3\gamma + d\rvert}{\sqrt{n_1^2 + n_2^2 + n_3^2}}$

The shortest distance between the two skew lines with equations $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ and $\boldsymbol{r} = \boldsymbol{c} + \mu\boldsymbol{d}$ where λ and μ are scalars	$\left\lvert\frac{(\boldsymbol{a} - \boldsymbol{c}).(\boldsymbol{b} \times \boldsymbol{d})}{\lvert(\boldsymbol{b} \times \boldsymbol{d})\rvert}\right\rvert$

Calculating Areas

Area of a triangle ABC where A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$	$\frac{1}{2}\lvert(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{b} \times \boldsymbol{c}) + (\boldsymbol{c} \times \boldsymbol{a})\rvert$
Area of a parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$	$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert$

Angle Between a Line and Plane

The angle θ between the line with equation $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ and the plane with the equation $\boldsymbol{r}.\boldsymbol{n} = p$ is given by:	$\sin\theta = \left\lvert\frac{\boldsymbol{b}.\boldsymbol{n}}{\lvert\boldsymbol{b}\rvert\lvert\boldsymbol{n}\rvert}\right\rvert$

Angle Between Two Planes

The angle θ between the plane with the equation $\boldsymbol{r}.\boldsymbol{n_1} = p_1$ and the plane with the equation $\boldsymbol{r}.\boldsymbol{n_2} = p_2$ is given by:	$\cos\theta = \left\lvert\frac{\boldsymbol{n_1}.\boldsymbol{n_2}}{\lvert\boldsymbol{n_1}\rvert\lvert\boldsymbol{n_2}\rvert}\right\rvert$

Polar Coordinates

For a point with polar coordinates (r, θ) and cartesian coordinates (x, y):	$r\cos\theta = x$ $r\sin\theta = y$
	$r^2 = x^2 + y^2$ $\theta = \arctan\left(\frac{y}{x}\right)$

$r = a$ is a circle with centre O and radius a
$\theta = \alpha$ is a half-line through O and making an angle α with the initial line
$r = a\theta$ is a spiral starting at O

Area of a Sector

The area of a sector bounded by a polar curve and the half lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians:	$\frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

Hyperbolic Fuctions

Definitions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\operatorname{cosech}^2 x = \coth^2 x - 1$$

Inverse Functions

$$\operatorname{arcosh} x = \ln\{x + \sqrt{x^2 - 1}\} \quad (x \geq 1)$$

$$\operatorname{arsinh} x = \ln\{x + \sqrt{x^2 + 1}\}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \quad (|x| < 1)$$

Differential Equations

First Order Differential Equations

Integrating factor	$e^{\int P(x)dx}$

General Solution to Second-Order Differential Equations

The nature of the roots α and β of the auxiliary equation determine the general solution to the second-order differential equation $a\frac{d^2y}{dx^2} + \frac{bdy}{dx} + c = 0$:

Case 1: $b^2 > 4ac$	The general solution will be in the form $y = Ae^{\alpha x} + Be^{\beta x}$ where α and β are two real roots of the auxiliary equation.
Case 2: $b^2 = 4ac$	The general solution will be in the form $y = (A + Bx)e^{\alpha x}$ where α is a repeated root of the auxiliary equation.
Case 3: $b^2 < 4ac$	The general solution will be in the form $y = e^{px}(A\cos qx + B\sin qx)$ where the auxiliary equation has two complex conjugate roots $\alpha, \beta = p \pm qi$.

Particular Integral

$$\boldsymbol{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)}$$

Form of $f(x)$	Form of particular integral
p	λ
$p + qx$	$\lambda + \mu x$
$p + qx + rx^2$	$\lambda + \mu x + \nu x^2$
pe^{kx}	λe^{kx}
$p\cos\omega x + q\sin\omega x$	$\lambda\cos\omega x + \mu\sin\omega x$