Modelling with Differential Equations Cheat Sheet

This chapter focuses on applying the methods from the previous chapter to real-life scenarios. You will learn to form differential equations from a given scenario and use the solution of a differential equation to analyse the motion of a particle. We will then learn about simple, damped and forced harmonic motion and their relation to differential equations. Finally, we will learn to solve coupled first-order differential equations.

Modelling with first-order differential equations

You need to be comfortable solving first-order differential equations given in context and using your solutions to answer questions about the model. First-order differential equations are often used to model problems in kinematics. Here is an example:

Example 1: A sports car moves along a horizontal straight road. At time t seconds, the acceleration, in ms^{-2}, is modelled using the differential equation $\frac{dv}{dt} - 2vt = t$, where v is the velocity of the car in ms^{-1}. When $t = 0$, the car is travelling at $1ms^{-1}$. Find the velocity of the car after 2 seconds.

This is a first-order differential equation. We want to find the velocity after $2s$ which means we first need to solve it.	$\frac{dv}{dt} - 2vt = t$
Use the integrating factor method since this DE is non-separable.	$I = e^{\int -2t\,dt} = e^{-t^2}$ $\Rightarrow e^{-t^2}\frac{dv}{dt} - 2e^{-t^2}vt = te^{-t^2}$
Use the product rule to simplify the LHS.	$\frac{d}{dt}(ve^{-t^2}) = te^{-t^2}$
Integrate both sides.	$ve^{-t^2} = \int te^{-t^2}\,dt$
To integrate the RHS, you can use by parts or notice that $\frac{d}{dt}\left(-\frac{1}{2}e^{-t^2}\right) = te^{-t^2}$.	$ve^{-t^2} = -\frac{1}{2}e^{-t^2} + c$
Multiply through by e^{t^2}.	$v = -\frac{1}{2} + ce^{t^2}$
Use the given initial condition $t = 0, v = 1$ to find c.	$1 = -\frac{1}{2} + c \therefore c = \frac{3}{2}$
So, our solution is:	$v = -\frac{1}{2} + \frac{3}{2}e^{t^2}$
To find the velocity after 2 seconds, substitute $t = 2$.	$v = -\frac{1}{2} + \frac{3}{2}e^4 = 81.4ms^{-1}$

As you can see, the methods are the same as in the previous chapter, Methods in Differential Equations. You just need to be comfortable in their application to problems in context.

Forming first-order differential equations

A problem that typically appears is one where you are expected to form a first-order differential equation using contextual information given to you in the question. These questions are similar in nature and often revolve around the rate of change of volume of a given quantity, where the quantity is both flowing in and out of a confined space.

Example 2: A gas storage tank initially contains 500 cm^3 of helium. The helium leaks out at a constant rate of 20 cm^3 per hour and a gas mixture is added to the tank at a constant rate of 50 cm^3 per hour. The gas mixture contains 5% oxygen and 95% helium. Given that there is $x\text{ cm}^3$ of oxygen in the tank after t hours and that the oxygen immediately mixes throughout the tank on entry, show that the situation can be modelled by the differential equation $\frac{dx}{dt} = 2.5 - \frac{2x}{50+3t}$.

$\frac{dx}{dt}$ represents the rate of change of oxygen. The amount of oxygen coming in per hour is $\frac{5}{100} \times 50 = \frac{5}{2}$.	$\left(\frac{dx}{dt}\right)_{in} = \frac{5}{100} \times 50 = \frac{5}{2}$
Amount of oxygen coming out per hour $= 20 \times (ratio\ of\ oxygen\ to\ total\ gas)$. The oxygen in the tank is given by x, and the total gas in the tank is given by $500 + 50t - 20t$ because there is initially $500\ cm^3$ of helium, $50cm^3$ is added every hour and $20\ cm^3$ is leaking every hour.	$\left(\frac{dx}{dt}\right)_{out} = 20 \times \frac{x}{500 + 50t - 20t}$ $= \frac{20x}{500 + 30t} = \frac{2x}{5 + 3t}$
The net rate of change of oxygen, $\frac{dx}{dt}$, is given by $\left(\frac{dx}{dt}\right)_{in} - \left(\frac{dx}{dt}\right)_{out}$.	$\therefore \frac{dx}{dt} = \frac{5}{2} - \frac{2x}{5 + 3t}$

By looking at the amount of gas flowing in separately to the amount of gas flowing out, we are able to break down the problem and use a logical approach.

Simple harmonic motion

Second order differential equations can be used to model particles moving with simple harmonic motion. You may be asked to solve these differential equations and use your solutions to answer questions about the model.

- Simple harmonic motion refers to any motion where the acceleration of a particle is always directed towards a fixed-point O and where the acceleration of the particle is proportional to its displacement from O.
- The motion of a particle moving with S.H.M will satisfy the differential equation $\ddot{x} = -\omega^2 x$, where ω is a constant. This is a second-order homogeneous differential equation.
- A particle moving with S.H.M will have zero velocity at the points of maximum displacement and maximum velocity when passing through the fixed point O.

The following representation of acceleration is important in proving general statements about simple harmonic motion:

- $\ddot{x} = v\frac{dv}{dx}$

Example 3: A particle P moves with simple harmonic motion about a point O. Given that the maximum displacement of the particle from O is a, show that $v^2 = \omega^2(a^2 - x^2)$.

We start using the two facts above:	$\ddot{x} = -\omega^2 x, \quad \ddot{x} = v\frac{dv}{dx}$
Equate the expressions for $\ddot{x}$. This is a separable differential equation.	$v\frac{dv}{dx} = -\omega^2 x$
Separate the variables.	$\int v\,dv = \int -\omega^2 x\ dx$
Carry out the integration.	$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + c$
We are told the max displacement from O is a. This means that at $x = a, v = 0$. Use this to find c.	$0 = -\frac{\omega^2 a^2}{2} + c \ \therefore\ c = \frac{\omega^2 a^2}{2}$
Our equation becomes:	$\frac{v^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{\omega^2 a^2}{2}$
Simplify the equation.	$v^2 = \omega^2(a^2 - x^2)$

Damped harmonic motion

In reality, there is likely to be an additional force (e.g. air resistance) that acts on a particle, which is proportional to the velocity of the particle. When this force causes the particle to slow down, it is known as a damping force and the particle is said to be moving with damped harmonic motion.

- For a particle moving with damped harmonic motion, $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = 0$ applies.

This is again a second-order homogeneous differential equation. The type of damping that the particle exhibits can be deduced from the roots of the auxiliary equation:

- When $k^2 > 4\omega^2$, the auxiliary equation has two distinct real roots and the particle is subject to heavy damping. There are no oscillations performed.
- When $k^2 = 4\omega^2$, the auxiliary equation has equal roots and the particle is subject to critical damping. There are no oscillations performed.
- When $k^2 < 4\omega^2$, the auxiliary equation has complex roots and the particle is subject to light damping. The amplitude of the oscillations will exponentially decay over time.

It is important you remember the above points, so you are able to determine the type of damping a particle is experiencing.

Example 4: A particle P hangs freely in equilibrium attached to one end of a light elastic string. The other end of the string is attached to a fixed point A. The particle is now pulled down and held at rest in a container of liquid which exerts a resistance to motion on P. P is then released from rest. While the string remains taut and the particle in the liquid, the motion can be modelled using the equation

$$\frac{d^2x}{dt^2} + 6k\frac{dx}{dt} + 5k^2x = 0\text{, where } k \text{ is a positive real constant.}$$

Find the general solution to the differential equation and state the type of damping the particle is subject to.

We solve the auxiliary equation using the quadratic formula.	$m^2 + 6km + 5k^2 = 0$ $m = -5k, m = k$
We found two real roots, so the particle is subject to heavy damping. Find the general solution.	$x = Ae^{-5kt} + Be^{-kt}$, the particle experiences heavy damping since we have two real roots.

Forced harmonic motion

When a particle is also subject to an external force that causes it to oscillate at a frequency other than its natural one, we say that the particle is moving with forced harmonic motion. The differential equation that consequently forms is no longer homogeneous.

- For a particle moving with forced harmonic motion, $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \omega^2 x = f(t)$ applies.

The principle is the same as before: we solve the differential equation using methods from the previous chapter and use our solution to answer any further questions about the model. The only difference here is that we will need to make use of the particular integral to find the complete solution.

Example 5: A particle P is attached to end A of a light elastic string AB. Initially the particle and string lie at rest on a smooth horizontal plane. At time $t = 0$, the end B of the string is set in motion and moves with constant speed U in the direction AB, and the displacement of P from A is x. Air resistance acting on P is proportional to its speed.

The subsequent motion can be modelled by the differential equation $\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + k^2x = 2kU$.
Find an expression for x in terms of U, k and t.

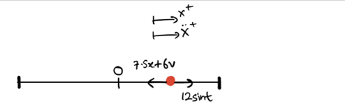
We first solve the corresponding auxiliary equation:	$m^2 + 2km + k^2 = 0$ $m = -k$ (repeated root)
We found two equal roots, so our complementary function is:	$x = (A + Bt)e^{-kt}$,
Now we need to find the particular integral. The RHS is $2kU$ so we choose our P.I to be a constant.	Let P.I.$= x = C$ $\dot{x} = \ddot{x} = 0$
Substitute our P.I. and its derivatives back into the original differential equation to find C.	$k^2C = 2kU$ $\therefore C = \frac{2U}{k}$
General solution = complementary function + particular integral	$\Rightarrow x = (A + Bt)e^{-kt} + \frac{2U}{k}$
To find the constants A and B, we need to use the given information in the question: at time $t = 0, x = 0$	$0 = A + \frac{2U}{k}$ $\therefore A = -\frac{2U}{k}$
We are also told that $v = 0$ at $t = 0$. We find v by differentiating x with respect to time.	$\dot{x} = v = -kAe^{-kt} + Be^{-kt} - kBte^{-kt}$
At t=0, v=U:	$U = -kA + B$ $B = U + kA = U + k\left(-\frac{2U}{k}\right) = -U$
Find the final form of the general solution.	$x = \left(-\frac{2U}{k} - Ut\right)e^{-kt} + \frac{2U}{k}$

Forming second-order differential equations relating to damped and forced harmonic motion

You may be asked to form a second-order differential equation from a given context. You will be told which forces act on a particle, but it's up to you to prove that the resulting equation of motion satisfies a given differential equation. To do so:

- Draw a diagram labelling all the forces acting on the particle and use $F = ma$ to obtain an equation of motion.
  - Acceleration should be taken as positive in the direction of increasing x.
  - Forces that tend to increase the displacement should be positive

Example 6: A particle P of mass 1.5kg is moving on the x-axis. At time t the displacement of P from the origin O is x metres and the speed of P is $v\ ms^{-1}$. Three forces act on P, namely a restoring force of magnitude 7.5x N, a resistance to the motion of P of magnitude 6v N and a force of magnitude 12sint N acting in the direction OP.
Show that $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8\sin t$.

We draw a diagram with all forces labelled. Acceleration is positive in the direction of x increasing. On our diagram, this is to the right.	
Using $F = ma$. The mass is 1.5kg. We can rewrite v as $\frac{dx}{dt}$ and $\ddot{x}$ as $\frac{d^2x}{dt^2}$.	$12sint - 7.5x - 6v = 1.5\ddot{x}$ $1.5\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 7.5x = 12sint$
Divide by 1.5	$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 8sint$ as required

Coupled first-order simultaneous differential equations

Finally, you need to be able to solve pairs of first order differential equations simultaneously.. You will be given a pair of first order differential equations of the form:

- $\frac{dx}{dt} = ax + by + f(t)$
- $\frac{dy}{dt} = cx + dy + g(t)$

Note that the letters x and y need not be used. The differential equations could be in terms of any two letters, c and d for example.

and you will be expected to form a second-order differential equation, in either x or y. To do so, you can use the process outlined in the below example:

Example 7: At the start of the year 2010, a survey began on the numbers of bears and fish on a remote island in Northern Canada. After t years the number of bears, b, and the number of fish, f, on the island are modelled by the differential equations: $\frac{db}{dt} = 0.3b + 0.1f$, $\frac{df}{dt} = -0.1b + 0.5f$. Show that $\frac{d^2b}{dt^2} - 0.8\frac{db}{dt} + 0.16b = 0$

We look at the equation we want to show, and notice that there is no f. So we want to eliminate f. To eliminate f, we look at the equation where there is only one term in f - this is the first equation.	$\frac{db}{dt} = 0.3b + 0.1f$
Using this first equation, make f the subject.	$f = 10\frac{db}{dt} - 3b$
Differentiate both sides with respect to t.	$\frac{df}{dt} = 10\frac{d^2b}{dt^2} - 3\frac{db}{dt}$
Substitute this result into the second given equation.	$10\frac{d^2b}{dt^2} - 3\frac{db}{dt} = -0.1b + 0.5f$
We still have a term in f. To get rid of it, we use the result from the second step: $f = 10\frac{db}{dt} + 3b$	$10\frac{d^2b}{dt^2} - 3\frac{db}{dt} = -0.1b + 0.5\left(10\frac{db}{dt} - 3b\right)$
Rearrange to give the required result.	$10\frac{d^2b}{dt^2} - 8\frac{db}{dt} + 1.6b = 0$ $\frac{d^2b}{dt^2} - 0.8\frac{db}{dt} + 0.16b = 0$