Methods in Differential Equations Cheat Sheet

Edexcel Core Pure 2

A differential equation is simply any equation involving derivatives. They occur surprisingly frequently in the real world. For example, think back to the constant acceleration formulae you encountered in Mechanics Year 1; we always modelled a given particle to be moving under the influence of gravity alone. However, what if we want a more accurate model that takes into account air resistance which changes depending on the speed of the particle? The equation of motion that subsequently forms is a differential equation. If we want to analyse the behaviour of such a particle, we need to be able to solve the differential equation. This chapter focuses on the methods you can use to solve a small set of first and second order linear differential equations.

Important definitions

- A differential equation is any equation involving derivatives. (e.g. $\frac{dy}{dx} = 2x$)
- The solution to a differential equation is a function.
- The order of a differential equation is the order of the highest derivative present in the equation.
- The general solution of a differential equation is a solution given in terms of unknown constants,
- A particular solution of a differential equation is a solution involving no unknown constants. This is found by substituting boundary conditions into the general solution to find the unknown constants.

First-order differential equations

You will be given first-order differential equations that can be solved using one of two methods: Separation of Variables and the integrating factor method. It is crucial you are able to recognise when each method is to be used.

- If a first-order differential equation is of the form $\frac{dy}{dx} = f(x)g(y)$, then you can solve it using the Separation of Variables technique, which you covered in Chapter 11 of Pure Year 2:
  - When $\frac{dy}{dx} = f(x)g(y)$, we can write $\int \frac{1}{g(y)}dy = \int f(x)\,dx$
- If a first-order differential equation is of the form $\frac{dy}{dx} + P(x)y = Q(x)$, then you can solve it using the integrating factor method, where the integrating factor is $e^{\int P(x)\,dx}$.

The integrating factor method can be summarised in five steps:

1. Manipulate the equation into the form $\frac{dy}{dx} + P(x)y = Q(x)$. **(I)**
   Note that if the equation cannot be rearranged into this form then you cannot use this method.
2. Find the integrating factor $e^{\int P(x)\,dx}$, simplifying as much as possible. Note that when evaluating the integral in the exponent, you do not need to add the constant of integration.
3. Multiply the equation **(I)** by the integrating factor.
4. Use the product rule 'in reverse' to simplify the LHS.
5. Integrate both sides of the equation to attain the general solution.

When dealing with first order differential equations, you should first see if it is separable (i.e. check if it can be written in the form $\frac{dy}{dx} = f(x)g(y)$). If it can then you should proceed to solve the equation using Separation of Variables. Otherwise you will need to use the integrating factor method,

We will now go through two different examples, each using one of the above two methods.

Example 1: Find the general solution to the differential equation $\frac{dy}{dx} = -\frac{y}{x}$.

Rewrite the equation in the form $\frac{dy}{dx} = f(x)g(y)$.	$\frac{dy}{dx} = -(x)^{-1}(y)$
Use $\int \frac{1}{g(y)}dy = \int f(x)\,dx$.	$\int \frac{1}{y}dy = -\int \frac{1}{x}dx$
Integrate both sides and add the constant of integration c.	$\ln\lvert y\rvert = -\ln\lvert x\rvert + c$ $\ln\lvert y\rvert + \ln\lvert x\rvert = c$
Use the multiplicative log law.	$\ln\lvert xy\rvert = c$
Use the rule: $\log_a b = x \Rightarrow a^x = b$. Since e^c is just a constant, we can rename it A for simplicity.	$xy = e^c = A$ $xy = A$
Divide through by x.	$\therefore y = \frac{A}{x}$

Example 2: Find the general solution to the differential equation $\cos x\frac{dy}{dx} + 2y\sin x = \cos^4 x$.

First, we check if we could use Separation of Variables.	This equation cannot be rearranged into the form $\frac{dy}{dx} = f(x)g(y)$, so we must use the integrating factor method.
This is not currently in the form $\frac{dy}{dx} + P(x)y = Q(x)$, so we need to get it into this form. Divide through by $\cos x$.	$\frac{dy}{dx} + 2y\tan x = \cos^3 x \quad (I)$
This is now in the required form. Find the integrating factor I.F.	$I.F = e^{\int 2\tan x\,dx} = e^{2\ln\lvert\sec x\rvert}$
Simplify the I.F. using the properties of ln .	$I.F. = e^{\ln\lvert\sec^2 x\rvert} = \sec^2 x$
Multiply equation (I) by $sec^2 x$.	$\sec^2 x\frac{dy}{dx} + 2y\sec^2 x\tan x = \sec^2 x\cos^3 x$
Simplify the RHS.	$\sec^2 x\frac{dy}{dx} + 2y\sec^2 x\tan x = \cos x$
Now, we use the product rule. Notice that the LHS is simply the derivative of $y\sec^2 x$.	$LHS = \frac{d}{dx}(y\sec^2 x)$ $\frac{d}{dx}(y\sec^2 x) = \cos x$
Integrate both sides (with respect to x).	$y\sec^2 x = \int \cos x\ dx$
Carry out the integration.	$y\sec^2 x = \sin x + c$
Finally, make y the subject.	$y = \frac{\sin x + c}{\sec^2 x} = \cos^2 x(\sin x + c)$ $y = \cos^2 x(\sin x + c)$

You might at first find it difficult to spot the use of the product rule. A trick is to look at the term involving $\frac{dy}{dx}$: replacing the $\frac{dy}{dx}$ with y gives you the required expression on which the product rule is used. For example,

$$(\sec x + \tan x)\frac{dy}{dx} + (\sec^2 x(1+\tan x))y = \sec^2 x(1+\tan x) \longrightarrow \frac{d}{dx}((\sec x + \tan x)y) = \sec x(1+\tan x)$$

$$(x^2+1)\frac{dy}{dx} + 2xy = 2x(x^2+1)^2 \longrightarrow \frac{d}{dx}((x^2+1)y) = 2x(x^2+1)^2$$

The terms involving $\frac{dy}{dx}$ are in green, while the expression on which the product rule is applied is in red. Notice that the only difference between the red and green terms is that the $\frac{dy}{dx}$ is replaced with y.

Second-order homogeneous differential equations

Equations of the form $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b and c are real constants, are known as second-order homogeneous differential equations. You need to be able to solve such differential equations.

- To solve a second-order homogeneous differential equation, you need to consider the auxiliary equation. The auxiliary equation of $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ will be $am^2 + bm + c = 0$.
- The nature of the solutions to the auxiliary equation will determine the general solution. You can use the quadratic formula (or another means) to find the roots of the auxiliary equation. There are three cases you need to consider:
- **Case 1:** $b^2 - 4ac > 0$
  - The auxiliary equation has two real roots, α and β. The general solution will be:
    $y = Ae^{\alpha x} + Be^{\beta x}$
- **Case 2:** $b^2 - 4ac = 0$
  - The auxiliary equation has one repeated root, α. The general solution will be:
    $y = e^{\alpha x}(A + Bx)$
- **Case 3:** $b^2 - 4ac < 0$
  - The auxiliary equation has two complex conjugate roots $p \pm iq$. The general solution will be:
    $y = e^{px}(A\sin(qx) + B\cos(qx))$

Example 3: Find the general solution to the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 34y = 0$.

We first form the auxiliary equation. Here, $a = 1, b = -6, c = 34$.	$m^2 - 6m + 34 = 0$
Use the quadratic formula to solve.	$m = \frac{6 \pm \sqrt{-100}}{2}$ $m = 3 \pm 5i$
This is an example of case 3.	$y = e^{3x}(A\sin(5x) + B\cos(5x))$

Second-order non-homogeneous differential equations

Equations of the form $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$, where a, b and c are real constants, are known as second-order non-homogeneous differential equations. The difference between homogenous and non-homogeneous differential equations is that rather than having 0 on the RHS, we have a function of x.

The process of solving such differential equations can be summarised in four steps:

1. Find the general solution to the corresponding homogeneous differential equation, $a\frac{dy}{dx} + b\frac{dy}{dx} + cy = 0$. This solution is known as the complementary function ($C.F.$)
2. Next, you need to find the particular integral ($P.I.$). This is a function that satisfies the original differential equation. The form of the particular integral will depend on the form of $f(x)$, the RHS of the original differential equation. You can use the following table to determine the form your $P.I.$ should take:

Form of $f(x)$	Form of $P.I.$
p	λ
$p + qx$	$\lambda + \mu x$
$p + qx + rx^2$	$\lambda + \mu x + \varphi x^2$
pe^{kx}	λe^{kx}
$p\cos(\omega x) + q\sin(\omega x)$	$\lambda\cos(\omega x) + \mu\sin(\omega x)$

3. Once you have determined the form of the $P.I.$, you need to substitute it back into the original differential equation to find any unknown constants in your $P.I.$ You will need to differentiate your $P.I.$ twice.
4. The general solution to the original differential equation is given by: $y = C.F. + P.I.$

Example 4: Find the general solution to the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 25x^2 - 7$.

We first find the complementary function by considering the homogeneous equation. Solve the auxiliary equation.	$m^2 - 4m + 5 = 0$ $m = 2 \pm i$ $\therefore C.F. \Rightarrow y = e^{2x}(A\sin x + B\cos x)$
Now we look at the form of $f(x)$ to determine the form our $P.I.$ should be in.	$f(x) = 25x^2 - 7.$ $P.I. \Rightarrow y = \lambda + \mu x + \varphi x^2$
Next, we want to substitute our $P.I.$ back into the original equation. To do so, we first need to find $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ for our $P.I.$	$\frac{dy}{dx} = 2\varphi x + \mu$ $\frac{d^2y}{dx^2} = 2\varphi$
Substitute into our original equation.	$(2\varphi) - 4(2\varphi x + \mu) + 5(\lambda + \mu x + \varphi x^2)$ $= 25x^2 - 7$
Compare coefficients of x^2, x and constants to find the values of λ, μ and φ.	x^2: $5\varphi = 25$ x: $8\varphi + 5\mu = 0$ $Constants$: $2\varphi - 8\mu + 5\lambda = 0$ Solving the above system of equations gives: $\varphi = 5, \mu = 8, \lambda = 3$
Deduce the particular integral, $P.I.$	$P.I. \Rightarrow y = 3 + 8x + 5x^2$
General solution $y = C.F. + P.I.$	$\Rightarrow y = e^{2x}(A\sin x + B\cos x) + 3 + 8x + 5x^2$

Using boundary conditions

You can be expected to use boundary conditions to find the particular solution of a differential equation. A boundary condition essentially just tells you a point that your solution passes through, which allows you to find an unknown constant in your general solution. Once you have found all unknowns in your general solution, the resulting solution is known as the particular solution.

- General solutions to first-order differential equations will have one unknown, whereas for second-order differential equations there will be two unknowns. This means that to find the particular solution to a first-order equation you will need one boundary condition whereas for a second-order equation you will need two boundary conditions.

Example 5: Given that the general solution to the differential equation $\frac{d^2y}{dx^2} + 9y = 16\sin x$ is $y = A\cos 3x + B\sin 3x + 2\sin x$, find the particular solution that satisfies $y = 1$ and $\frac{dy}{dx} = 8$ at $x = 0$.

Substitute $x = 0, y = 1$ into the general solution.	$1 = A\cos 0 + B\sin 0 + 2\sin 0 \Rightarrow A = 1$
Differentiate to find $\frac{dy}{dx}$ then substituting $x = 0$, $\frac{dy}{dx} = 8$.	$\frac{dy}{dx} = -3A\sin 3x + 3B\cos 3x + 2\cos x$ $8 = 3B + 2 \Rightarrow B = 2$
Substitute $A = 1, B = 2$ into the general solution.	$\therefore y = \cos 3x + 2\sin 3x + 2\sin x$