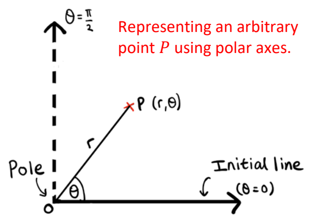
Polar Coordinates Cheat Sheet

Edexcel Pure Year 2

You are likely very used to describing the position of a point in two-dimensions using x and y coordinates. We will now look at an alternative system to describe the position of such a point, known as Polar Coordinates, where we instead use angles and distances to describe positions. An example of where this system is superior to the xy system can be found in mechanics; the equation of motion for a particle moving in circular motion is greatly simplified when working with polar coordinates, as opposed to cartesian coordinates.

Using polar coordinates

We need two measurements, r and θ, to describe a point, P, in polar coordinates.

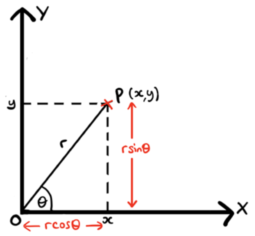


- r is the distance of the point P from the pole (the origin O).
- θ is the angle measured anticlockwise between the initial line and the line OP.
- Polar coordinates are given in the form (r, θ).

Note that when sketching polar curves, it is conventional to draw the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ instead of the x and y axes. Keep in mind that not all sketches will have these lines labelled, however.

Converting between cartesian and polar forms

To understand the relationship between cartesian and polar coordinates, let's take a close look at the following diagram:



We can make a few deductions about the relationship between cartesian and polar coordinates from this diagram:

- $r\cos\theta = x$
- $r\sin\theta = y$

$\Rightarrow$

- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$

You need to be comfortable using the above four bullet points to convert between the cartesian and polar forms for coordinates and equations.

Example 1: Find the polar equation for the curve with cartesian equation $x^2 - y^2 = 5$.

Using $x = r\cos\theta$, $y = r\sin\theta$	$r^2\cos^2\theta - r^2\sin^2\theta = 5$
Factorise out r^2	$r^2(\cos^2\theta - \sin^2\theta) = 5$
Use the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$	$r^2(\cos 2\theta) = 5$
Make r^2 the subject	$r^2 = \frac{5}{\cos 2\theta}$ $\therefore r^2 = 5\sec 2\theta$

You can also be expected to convert a locus of points on an Argand diagram into polar form.

- To convert complex loci into polar form, you should first convert into cartesian form, then into polar form.

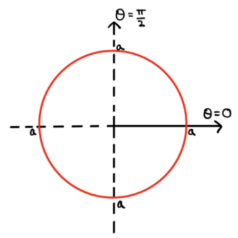
Example 2: Show that the locus of points given by the values of z satisfying $|z - 3 - 4i| = 5$ can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$.

Convert the locus into cartesian form. (See chapter 2 of Core Pure 1)	This is simply a circle with centre (3,4) and radius 5 $\Rightarrow (x-3)^2 + (y-4)^2 = 25$
Use $x = r\cos\theta$, $y = r\sin\theta$ to now convert into polar form	$(r\cos\theta - 3)^2 + (r\sin\theta - 4)^2 = 25$ $r^2\cos^2\theta + r^2\sin^2\theta - 6r\cos\theta - 8r\sin\theta = 25 - 25$
Use $\sin^2\theta + \cos^2\theta = 1$	$r^2(1) = 6r\cos\theta + 8r\sin\theta$
Divide through by r	$r = 6\cos\theta + 8\sin\theta$ as required.

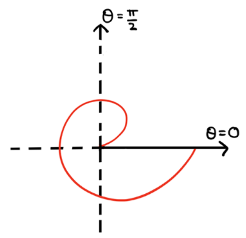
Sketching simple polar curves

You need to be able to sketch a small set of simple polar curves, by learning their general shapes.

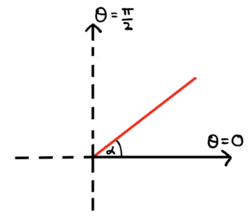
$r = a$ represents a circle with centre O and radius a.



$r = a\theta$ represents a spiral beginning at O.



$\theta = \alpha$ represents a half-line that makes an angle α with the initial line and passes through O.



Sketching harder polar curves

You also need to be able to sketch polar curves involving sine and cosine functions. For such curves, it is a good idea to construct a table of key values of θ for positive values of r to help with your sketch. Here are some facts that are helpful to remember when sketching such curves:

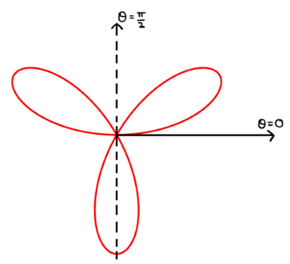
- Equations of the form $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$ will have n loops symmetrically arranged around the pole (see example 3).
- Curves of the form $r = a(p + q\cos\theta)$, where $p, q > 0$, will:
  - be circular when $p = 0$.
  - be a cardioid when p = q.
  - be egg-shaped when $\frac{p}{q} \geq 2$.
  - have a "dimple" when $1 \leq \frac{p}{q} < 2$.

If you have a graphical calculator then it is a good idea to use it to help with your sketches.

Example 3: Sketch the curve with equation $r = a\sin 3\theta$, where a is a positive constant.

Looking at the equation, we know that this curve will have 3 loops arranged symmetrically around O. The values of θ we need to consider are those for which r is positive. This is where $0 \leq \theta \leq \frac{\pi}{3}$, $\frac{2\pi}{3} \leq \theta \leq \pi$ and $\frac{4\pi}{3} \leq \theta \leq \frac{5\pi}{3}$. We deduce these ranges by looking at where $\sin 3\theta$ is positive. We then consider one loop (i.e. one of the aforementioned ranges), construct a table and use it to sketch the rest of the curve:

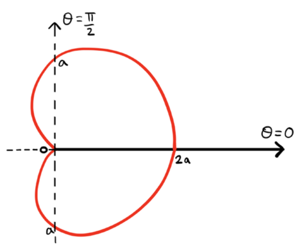
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
r	0	a	0



Example 4: Sketch the curve with equation $r = a(1 + \cos\theta)$, where a is a positive constant.

Note that $\frac{p}{q} = \frac{a}{a} = 1 < 2$, so this curve will have a 'dimple'. As p = q, the curve that will be drawn is called a cardioid. We now construct a table of values to help us with the sketch, using $\theta = 0, \frac{\pi}{2}, \pi, \frac{\pi}{2}$ and 2π. These values are enough to sketch the curve.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	a	0	a	$2a$

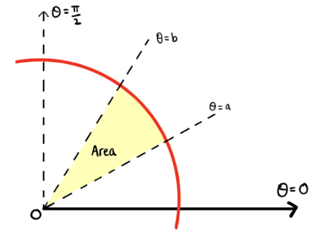


Finding the area enclosed by a polar curve

You also need to be able to use integration to find the area enclosed by a polar curve.

To find the area of a sector enclosed by a polar curve and the half-lines $\theta = a$ and $\theta = b$, you can use the formula:

- $$Area = \frac{1}{2}\int_a^b r^2 \, d\theta$$



You will need to use your knowledge of integration from Pure Year 2 to evaluate polar integrals. Knowledge of the following double angle cosine identities are especially important, since integrals will often contain $sin^2\theta$ and $cos^2\theta$ terms.

- $$\cos(2\theta) \equiv \cos^2(\theta) - \sin^2(\theta) \equiv 2\cos^2(\theta) - 1 \equiv 1 - 2\sin^2(\theta)$$

Harder problems will require you to find an area that is enclosed by more than one curve. The principle is the same; you should use the above formula, but you may need to evaluate more than one integral to find the required area.

Tangents to polar curves

Finally, you need to be able to find the tangent to a polar curve that is either parallel or perpendicular to the initial line.

- A tangent will be parallel to the initial line when $\frac{dy}{d\theta} = 0$.
- A tangent will be perpendicular to the initial line when $\frac{dx}{d\theta} = 0$.

Rather than memorising each case, a better way of figuring out what to set equal to 0 is to think about the orientation of the tangent:

When the tangent is parallel to the initial line, it is horizontal and therefore along the line there is no change in y. So, we set $\frac{dy}{d\theta} = 0$. When the tangent is perpendicular to the initial line however, it is vertical and therefore along the line there is no change in x. So, we set $\frac{dx}{d\theta} = 0$ instead.

Example 5: Find the points on the curve $r = a\cos 2\theta$, $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$, where the tangents are parallel to the initial line, giving your answer to 3 $s.f.$ where appropriate.

Recall that $\frac{dy}{d\theta} = 0$ when the tangent is parallel to the initial line. So, we need to get y into our equation. To do so, we can multiply both sides by $\sin\theta$.	$r\sin\theta = a\sin\theta\cos 2\theta$ $\therefore y = a\sin\theta\cos 2\theta$
Find $\frac{dy}{d\theta}$ using the product rule and setting equal to 0. Divide through by a.	$\frac{dy}{d\theta} = a\cos\theta\cos 2\theta - 2a\sin\theta\sin 2\theta = 0$ $\therefore \cos\theta\cos 2\theta - 2\sin\theta\sin 2\theta = 0$
To solve this equation, we can first use the double angle formula $\sin 2\theta = 2\sin\theta\cos\theta$	$\cos\theta\cos 2\theta - 4\sin^2\theta\cos\theta = 0$ $\cos\theta(\cos 2\theta - 4\sin^2\theta) = 0$
We have two sets of solutions: one corresponds to when $\cos\theta = 0$ and the other when $(\cos 2\theta - 4\sin^2\theta) = 0$. However, there are no solutions for $\cos\theta = 0$ in the given range so we solve the other equation. We use the double angle formula $\cos 2\theta \equiv 1 - 2\sin^2(\theta)$ here:	$\cos\theta = 0 \Rightarrow no\ solutions.$ $\cos 2\theta - 4\sin^2\theta = 0$ $1 - 2\sin^2\theta - 4\sin^2\theta = 0$ $\therefore \sin^2\theta = \frac{1}{6} \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{6}}$
Using CAST or a graphical method, we can solve for the values of θ in the given range:	$\Rightarrow \theta = \pm 0.421$
Finally, we find the corresponding value of r and then writing the coordinates. Note that r is the same for $\theta = \pm 0.421$ since $\cos(0.421) = \cos(-0.421)$	$r = acos2\theta = a\cos 2(\pm 0.421) = \frac{2a}{3}$ So our coordinates are: $\left(\frac{2a}{3}, \pm 0.421\right)$