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Volumes of Revolution Cheat Sheet

Edexcel Core Pure 2

Previously in Core Pure 1, you learnt how to find volumes of revolutions for simpler functions, usually polynomials. We will now extend this further and learn to do so for more complex functions. You will need to be very proficient with the methods of integration covered in Pure Year 2 to be successful with problems relating to this chapter.

Volumes of revolution around the x-axis

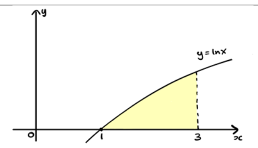
Recall from Core Pure 1:

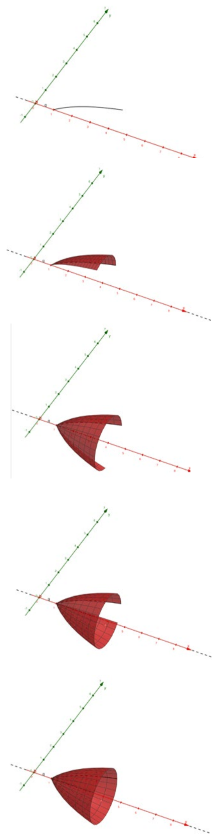
- The volume of revolution, V, formed when a function $y = f(x)$ is rotated through 2π radians about the x-axis between the lines $x = a$ and $x = b$ is given by

$$V = \pi \int_a^b y^2 \, dx$$

Here is an example illustrating how we find the volume of revolution formed by a more complicated curve being rotated around the x-axis:

Example 1: The finite region R represents the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 3$. The region R is rotated through 2π radians about the x-axis. Use integration to find the exact volume of the solid generated.

We start with a quick sketch to clarify the region represented by R. Since $y = \ln x$ cuts the x-axis at $x = 1$, our limits of integration are $x = 1, x = 3$.	
Using the above formula:	$V = \pi \int_a^b y^2 \, dx = \pi \int_1^3 (\ln x)^2 \, dx$
To evaluate an integral of the form $[\ln(x)]^n$, it is a good idea to use the 'by parts' method.	$\frac{dv}{dx} = 1, \quad u = (\ln x)^2$ $\therefore v = x, \quad \frac{du}{dx} = 2(\ln x)\left(\frac{1}{x}\right)$
Substituting into the 'by parts' formula:	$\therefore \int_1^3 (\ln x)^2 \, dx = [x(\ln x)^2]_1^3 - 2\int_1^3 \ln x \, dx$ $= 3(\ln 3)^2 - 2\int_1^3 \ln x \, dx$
To evaluate $\int_1^3 \ln x \, dx$ we need to use by parts again:	$\frac{dv}{dx} = 1, \quad u = \ln x$ $\therefore v = x, \quad \frac{du}{dx} = \left(\frac{1}{x}\right)$
Substituting into the by parts formula:	$\therefore \int_1^3 \ln x \, dx = [x \ln x]_1^3 - \int_1^3 1 \, dx$ $= 3\ln 3 - 2$
Back substituting our result to find $\int_1^3 (\ln x)^2 \, dx$:	$\therefore \pi \int_1^3 (\ln x)^2 \, dx = \pi[3(\ln 3)^2 - 2(3\ln 3 - 2)]$ $= \pi[3(\ln 3)^2 - 6\ln 3 + 4]$



Volumes of revolution around the y-axis

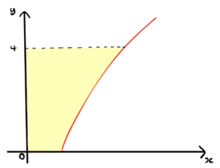
Recall from Core Pure 1:

- The volume of revolution, V, formed when a function $y = f(x)$ is rotated through 2π radians about the y-axis between the lines $y = a$ and $y = b$ is given by

$$V = \pi \int_a^b x^2 \, dy$$

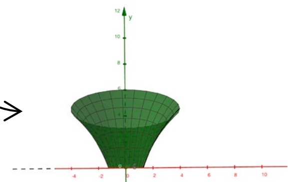
It is important to remember that when finding the volume of revolution around the y-axis, any integration is done with respect to y. You may have to rearrange a given equation to make x^2 the subject before you can find any volumes of revolution.

Example 2: The diagram shows the curve with equation $y = 4\ln x - 1$. The finite region R, shown in the diagram, is bounded by the curve, the x-axis, the y-axis and the line $y = 4$. Region R is rotated by 2π radians about the y-axis. Use integration to show that the exact value of the volume of the solid generated is $2\pi\sqrt{e}(e^2 - 1)$.



We first rearrange to make x the subject, then square the entire expression to attain x^2:	$y + 1 = 4\ln x$ $\frac{y+1}{4} = \ln x$ $\therefore x = e^{\frac{y+1}{4}} = e^{\frac{y}{4}}e^{\frac{1}{4}} \Rightarrow x^2 = e^{\frac{y}{2}}e^{\frac{1}{2}}$
Using the volume of revolution formula:	$V = \pi \int_0^4 e^{\frac{y}{2}}e^{\frac{1}{2}} \, dy$
Carrying out the integration and simplifying the result. The result $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$ was used here.	$V = \pi\left[2e^{\frac{y}{2}}e^{\frac{1}{2}}\right]_0^4$ $= \pi\left[2e^{\frac{4}{2}}e^{\frac{1}{2}} - 2e^0 e^{\frac{1}{2}}\right]$ $= 2\pi\sqrt{e}(e^2 - 1)$

Resultant volume of revolution



Volumes of revolution of parametrically defined curves

You can also be expected to find volumes of revolution for curves given in parametric form. The formula in such cases is simply an adjustment of the previous formulas by use of the chain rule:

- The volume of revolution, V, formed when the parametric curve with equations $x = f(t)$ and $x = g(t)$ is rotated through 2π radians about the x-axis between the lines $x = a$ and $x = b$ is given by

$$V = \pi \int_a^b y^2 \frac{dx}{dt} \, dt$$

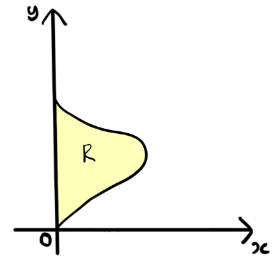
- The volume of revolution, V, formed when the parametric curve with equations $x = f(t)$ and $x = g(t)$ is rotated through 2π radians about the y-axis between the lines $y = a$ and $y = b$ is given by:

$$V = \pi \int_a^b x^2 \frac{dy}{dt} \, dt$$

It is important to remember that any integration should be done with respect to t. You may need to use the given parametrisation to find the limits of integration if they are not given to you in terms of t.

Example 3: The curve C is given by the parametric equations $x = \sin^4 t\sqrt{\cos t}, y = \cos t, 0 \le t \le \frac{\pi}{2}$. The finite region R bounded by the curve and the y-axis is rotated through 360° about the y-axis. Find the volume of the solid of revolution formed.

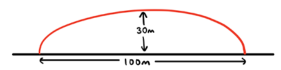
We need to find the quantities x^2 and $\frac{dy}{dt}$ and substitute them into the formula $V = \pi \int_a^b x^2 \frac{dy}{dt} \, dt$.	$x^2 = \sin^8 t \cos t$ $\frac{dy}{dt} = -\sin t$
Using the formula: We took the minus sign outside of the integral here.	$V = -\pi \int_0^{\frac{\pi}{2}} \sin^9 t \cos t \, dt$
To evaluate an integral, we use the reverse chain rule. Recall that $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$	$V = -\pi\left[\frac{\sin^{10} t}{10}\right]_0^{\frac{\pi}{2}}$ $= -\pi\left[\frac{1}{10}\right]$
Volume cannot be negative so we take the absolute value of our answer:	$V = \frac{\pi}{10}$



Modelling with volumes of revolution

You may also need to solve volumes of revolution problems involving a real-life scenario. Usually, there will be a symmetric object modelled by a curve and you will be required to find a volume enclosed by the object, either by rotating about the x or y axes. Sometimes, you will need to figure out which axes to rotate about and the limits of integration to be used. You may also need to use your knowledge of the shapes formed when particular lines are rotated:

Example 4: The diagram shows the cross-section of a domed tent. The tent can be modelled by a solid of revolution of a curve C about the y-axis. Curve C has parametric equations $x = 50cost. y = 30sint, 0 \le t \le \frac{\pi}{2}$. Find the volume of the tent.



We need to find the quantities x^2 and $\frac{dy}{dt}$ and substitute them into the formula $V = \pi \int_a^b x^2 \frac{dy}{dt} \, dt$.	$x^2 = 2500\cos^2 t$ $\frac{dy}{dt} = 30\cos t$
Using the formula, taking the 2500 outside the integral. You must notice that the limits 0 and $\frac{\pi}{2}$ given in the question correspond to only one half of the curve in the diagram. The required volume will therefore be found when we use the limits 0 and $\frac{\pi}{2}$.	$V = 75000\pi \int_0^{\frac{\pi}{2}} \cos^3 t \, dt$
To evaluate this integral, we rewrite it as a product of $\cos t$ and $\cos^2 t$. Then we use the identity $\cos^2 t \equiv 1 - \sin^2 t$.	$\int \cos^3 t \, dt = \int \cos^2 t \cos t \, dt$ $= \int (1 - \sin^2 t)\cos t \, dt$ $= \int \cos t \, dt - \int \sin^2 t \cos t \, dt$
We know from Pure Year 2 that $\cos t$ integrates to $\sin t$. To find the second integral, we used the result $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$.	$= \sin t - \frac{\sin^3 t}{3} (+c)$
Now we just need to apply the limits to our result.	$\therefore V = 75000\pi\left[\sin t - \frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}}$ $= 75000\pi\left[1 - \frac{1}{3}\right] = 75000\pi\left[\frac{2}{3}\right] = 50000\pi$

Resultant volume of revolution: