Complex Numbers Cheat Sheet

This chapter aims to build upon the complex numbers you learnt in Core Pure 1. We will look at Euler's formula and De Moivre's theorem; two powerful ideas which will lay the foundation for most of the techniques you will encounter in this chapter. Complex numbers themselves have an unexpectedly large number of applications in the real world, such as the modelling of quantum waves in Physics to the representation of alternating current in Electrical Engineering.

Exponential form of complex numbers

In Core Pure 1, you learnt that the modulus argument form of a complex number z is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\arg z = \theta$. You can use Euler's formula to express a complex number in an exponential form:

- $e^{i\theta} = \cos\theta + i\sin\theta$

So the complex number z can also be written as:

- $z = re^{i\theta}$, where $r = |z|$ and $\arg z = \theta$

This is the exponential form of a complex number. You need to be very comfortable expressing a complex number in both exponential and modulus-argument forms. The exponential form will be quite prevalent in this chapter.

The following results follow from Euler's formula and are worth remembering:

- $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$
- $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

These results are significant because they give us a direct connection between complex numbers and the trigonometric functions. You could be asked to prove these. The proof of the first statement is given in Example 2, and the proof for the second is very similar.

Example 1: Express the complex number $z = \sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ in the form $re^{i\theta}$.

We use the given form to figure out the modulus and argument of z	$\lvert z\rvert = \sqrt{2}, \quad \arg z = \frac{\pi}{2}$
Now using the exponential form	$\therefore z = \sqrt{2}e^{\frac{i\pi}{2}}$

Example 2: Use Euler's relation to show that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$.

Euler's relation states:	$e^{i\theta} = \cos\theta + i\sin\theta \quad (I)$
Replacing θ with $-\theta$. Note that $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$	$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$ $e^{-i\theta} = \cos(\theta) - i\sin(\theta) \quad (II)$
Subtracting (II) from (I):	$e^{i\theta} - e^{-i\theta} = \cos\theta - \cos\theta + i\sin\theta + i\sin\theta$ $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$
Dividing by $2i$	$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

Multiplying and dividing complex numbers

Recall from Core Pure 1 that for any two complex numbers z_1, z_2:

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

We can deduce similar results for when complex numbers are given in an exponential form:

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Example 3: Express $\sqrt{5}e^{i\theta} \times 3e^{3i\theta}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

The modulus of the resultant complex number is found by multiplying each modulus.	$\lvert z\rvert = \sqrt{5}, \lvert z\rvert = 3$ $\lvert z_1 z_2\rvert = 3\sqrt{5}$
The argument of the resultant complex number is found by adding the arguments together	$\arg z_1 = \theta, \arg z_2 = 3\theta$ $\arg(z_1 z_2) = 4\theta$
Using the modulus argument form to write the resultant complex number in the form $x + iy$:	$\therefore z_1 z_2 = 3\sqrt{5}(\cos(4\theta) + isin(4\theta))$ $= 3\sqrt{5}\cos(4\theta) + i(3\sqrt{5})sin(4\theta)$

De Moivre's theorem

You can use De Moivre's theorem to calculate powers of complex numbers:

- $(r(\cos\theta + i\sin\theta))^n = r^n(\cos(n\theta) + i\sin(n\theta))$

If we consider the exponential form, this result seems more obvious:

- $(re^{i\theta})^n = r^n e^{i(n\theta)}$

This formula allows you to easily simplify some seemingly complicated expressions, like the one in Example 4.

Example 4: Evaluate $\dfrac{(cos\frac{7\pi}{13} + isin\frac{7\pi}{13})^4}{(cos\frac{4\pi}{13} + isin\frac{4\pi}{13})^6}$ giving your answer in the form $x + iy$, where $x, y \in \mathbb{R}$.

Use De Moivre's theorem with the numerator:	$(\cos\frac{7\pi}{13} + i\sin\frac{7\pi}{13})^4 = \cos\frac{28\pi}{13} + i\sin\frac{28\pi}{13}$
Use De Moivre's theorem with the denominator:	$(\cos\frac{4\pi}{13} + i\sin\frac{4\pi}{13})^6 = \cos\frac{24\pi}{13} + i\sin\frac{24\pi}{13}$
So, the whole fraction simplifies to:	$\dfrac{\cos\frac{28\pi}{13} + i\sin\frac{28\pi}{13}}{\cos\frac{24\pi}{13} + i\sin\frac{24\pi}{13}}$
We can simplify this using the rule for dividing complex numbers: we divide the magnitudes and subtract the arguments.	$= \cos\left(\frac{28\pi}{13} - \frac{24\pi}{13}\right) + i\sin(\frac{28\pi}{13} - \frac{24\pi}{13})$ $= \cos\left(\frac{4\pi}{13}\right) + i\sin\left(\frac{4\pi}{13}\right)$

Trigonometric identities

You can also be expected to use De Moivre's theorem to derive trigonometric identities. The following results are important for such problems:

If $z = \cos\theta + i\sin\theta$, then

- $z^n + \frac{1}{z^n} = 2\cos n\theta$
- $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Proof: $z^n = \cos(n\theta) + i\sin(n\theta) \quad (I)$
$z^{-n} = \frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta) = \cos(n\theta) - i\sin(n\theta) \quad (II)$
Adding (I) and (II) gives:
$z^n + \frac{1}{z^n} = 2\cos(n\theta)$
To prove the second statement, we would instead subtract (II) from (I).

You could be asked to prove any of the above results. Examples 5 shows how you can use these results to prove trigonometric identities.

Example 5: Express $cos^5\theta$ in the form a $\cos(5\theta) + b\cos(3\theta) + c\cos(\theta)$, where a, b and c are constants.

Using $z^n + \frac{1}{z^n} = 2\cos n\theta$ with $n = 1$:	$z + \frac{1}{z} = 2\cos\theta$
Raising both sides to the fifth power:	$\left(z + \frac{1}{z}\right)^5 = 32\cos^5\theta$
We now focus on the LHS and expand using the binomial expansion:	$\left(z + \frac{1}{z}\right)^5 = z^5 + 5(z^4)\left(\frac{1}{z}\right) + 10(z^3)\left(\frac{1}{z^2}\right) + 10(z^2)\left(\frac{1}{z^3}\right) + 5(z)\left(\frac{1}{z^4}\right) + \frac{1}{z^5}$
We can pair up the terms that match in power:	$\left(z + \frac{1}{z}\right)^5 = \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$
These terms can all be simplified using: $z^n + \frac{1}{z^n} = 2cosn\theta$	$\left(z + \frac{1}{z}\right)^5 = 2\cos(5\theta) + 5(2\cos(3\theta)) + 10(2\cos(\theta))$ $= 2\cos(5\theta) + 10\cos(3\theta) + 20\cos(\theta)$
But from the second step we said that $\left(z + \frac{1}{z}\right)^5 = 32cos^5\theta$, so we can say that:	$32\cos^5\theta = 2\cos(5\theta) + 10\cos(3\theta) + 20\cos(\theta)$
Dividing both sides by 32:	$\cos^5\theta = \frac{1}{16}\cos(5\theta) + \frac{5}{16}\cos(3\theta) + \frac{5}{8}\cos(\theta)$

Sums of complex series

Recall from Chapter 3 of Pure Year 2 that for a geometric series:

- The sum of the first n terms is given by $S_n = \frac{a(1-r^n)}{1-r}$.
- The sum to infinity is given by $S_\infty = \frac{a}{1-r}$.

You can also use these results when a and r are complex. Questions involving series will often require a lot of algebraic manipulation to achieve the final result.

Example 6: The series P and Q are defined for $0 < \theta < \pi$ as $P = 1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos 12\theta + \cdots$
$Q = \sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin 12\theta + \cdots$

Show that $P + iQ = \dfrac{e^{6i\theta}\left(e^{\frac{13i\theta}{2}} - e^{-\frac{13i\theta}{2}}\right)}{2i\sin\left(\frac{\theta}{2}\right)}$

Adding P to iQ, we can see that we are dealing with a geometric series.	$P + iQ = 1 + (\cos\theta + i\sin\theta)$ $+ (\cos 2\theta + i\sin 2\theta) + \cdots$
We can use the previous line to figure out what a and r are for this geometric series. Using the exponential form where possible will make any manipulation a lot easier.	So $a = 1, r = \cos\theta + i\sin\theta = e^{i\theta}$
There are 13 terms in total (since the first term is 1), so using the sum of a geometric series formula with $n = 13$:	$P + iQ = \dfrac{1(1 - (e^{i\theta})^{13})}{1 - e^{-i\theta}} = \dfrac{1 - e^{13i\theta}}{1 - e^{i\theta}}$
We can rewrite $1 - e^{13i\theta}$ as $e^{\frac{13i\theta}{2}}(e^{\frac{-13i\theta}{2}} - e^{\frac{13i\theta}{2}})$ This is a common trick you often need to use for series questions.	$= \dfrac{e^{\frac{13i\theta}{2}}(e^{\frac{-13i\theta}{2}} - e^{\frac{13i\theta}{2}})}{1 - e^{i\theta}}$

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Notice that the result we want to show has $2i\sin\left(\frac{\theta}{2}\right)$ in the denominator. And recall that $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$, so $2i\sin\left(\frac{\theta}{2}\right) = e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}$ So if we multiply the top and bottom by $e^{-\frac{i\theta}{2}}$, we get $e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}$ on the bottom, which is equal to $-2i\sin\left(\frac{\theta}{2}\right)$	$= \dfrac{e^{\frac{-i\theta}{2}}e^{\frac{13i\theta}{2}}(e^{\frac{-13i\theta}{2}} - e^{\frac{13i\theta}{2}})}{e^{\frac{-i\theta}{2}}(1 - e^{i\theta})} = \dfrac{e^{6i\theta}(e^{\frac{-13i\theta}{2}} - e^{\frac{13i\theta}{2}})}{e^{\frac{-i\theta}{2}} - e^{\frac{i\theta}{2}}}$
The denominator is now equal to $-2i\sin\left(\frac{\theta}{2}\right)$. Multiplying the top and bottom by -1 gives us the required result.	$\therefore P + iQ = \dfrac{e^{6i\theta}(e^{\frac{-13i\theta}{2}} - e^{\frac{13i\theta}{2}})}{-2i\sin\left(\frac{\theta}{2}\right)} = \dfrac{e^{6i\theta}(e^{\frac{13i\theta}{2}} - e^{\frac{-13i\theta}{2}})}{2i\sin\left(\frac{\theta}{2}\right)}$

nth roots of a complex number

Finding the n roots of a complex number w is equivalent to solving the equation $z^n = w$.

- The equation $z^n = w$ has n distinct solutions (z and w are non-zero complex numbers, n is a positive integer).

We use De Moivre's theorem to find the roots of a complex number, along with the following fact:

- $z = r(\cos(\theta) + i\sin(\theta)) = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, where k is any integer.

To solve an equation of the form $z^n = w$, you should follow the process used in Example 7 below:

Example 7: Solve the equation $z^4 + 2i\sqrt{3} = 2$, expressing the roots in the form $r(\cos\theta + i\sin\theta)$.

We start by making z^4 the subject:	$z^4 = 2 - i(2\sqrt{3})$
Writing in modulus-argument form: (we could also use the exponential form)	$z^4 = 4(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right))$
Taking the fourth root of both sides:	$z = 4^{\frac{1}{4}}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)^{\frac{1}{4}}$
But remember that if we add on any multiple of 2π to the argument, this will also be a solution, so we add $2k\pi$ to the argument. Make sure to do this BEFORE you use De Moivre's theorem, which is the next step.	$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{3} + 2k\pi\right) + i\sin\left(-\frac{\pi}{3} + 2k\pi\right)\right)^{\frac{1}{4}}$
Simplifying the argument into one fraction makes further working slightly easier:	$z = \sqrt{2}\left(\cos\left(\frac{-\pi + 6k\pi}{3}\right) + i\sin\left(\frac{-\pi + 6k\pi}{3}\right)\right)^{\frac{1}{4}}$
Now applying De Moivre's theorem:	$z = \sqrt{2}\left(\cos\left(\frac{-\pi + 6k\pi}{12}\right) + i\sin\left(\frac{-\pi + 6k\pi}{12}\right)\right)$
There are four solutions in total. We use different values of k that result in the argument being in the range $-\pi < \theta \leq \pi$	$k = 0: z = \sqrt{2}\,(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right))$ $k = 1: z = \sqrt{2}(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right))$ $k = 2: z = \sqrt{2}(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right))$ $k = -1: z = \sqrt{2}\,(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right))$

Solving geometric problems

The roots of a complex number when plotted on an argand diagram form a polygon. You can use this idea to solve geometric problems.

- The n roots of a complex number z lie at the vertices of a regular n-gon which has its centre at O.

For example, the solutions to the equation $z^4 = 2 + i$ are the vertices of a square with centre O. We will now look at the roots of unity, which are useful for geometric problems:

- An nth root of unity is a solution to the equation $z^n = 1$.
- If you know one root of a complex number with n roots, then you can find the other roots by multiplying by an nth root of unity.
- An nth root of unity is given by $\omega = e^{\frac{2\pi i}{n}}$. For example, if a complex number has four roots then a 'fourth' root of unity is given by $\omega = e^{\frac{2\pi i}{4}}$.

Example 8: The point $P(\sqrt{3}, 1)$ lies at one vertex of an equilateral triangle. The centre of the triangle lies at the origin. Find the coordinates of the other vertices of the triangle.

This is an equilateral triangle, so the three vertices represent the three roots of a complex number. We are given one root:	One root is $z = \sqrt{3} + i$ In exponential form: $z = 2e^{\frac{i\pi}{6}}$
To find the other roots, we need to multiply by an nth root of unity. There are three roots here, so we call it a cube root of unity:	Cube root of unity $= e^{\frac{2\pi i}{3}}$
We multiply the original root by the root of unity two successive times to find the other two roots. Remember that the roots correspond to the vertices.	$z = 2e^{\frac{i\pi}{6}} \times e^{\frac{2\pi i}{3}} = 2e^{\frac{5\pi i}{6}} = -\sqrt{3} + i$ $z = 2e^{\frac{5\pi i}{6}} \times e^{\frac{2\pi i}{3}} = 2e^{\frac{9\pi i}{6}} = -2i$
We write our answers as coordinates:	$(-\sqrt{3}, 1)$ and $(0, -2)$ are our vertices.