Roots of Polynomials

Questions

Q1.

The cubic equation

$$z^3 - 3z^2 + z + 5 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

(Total for question = 5 marks)

Q2.

The cubic equation

$$3x^3 + x^2 - 4x + 1 = 0$$

has roots α, β, and γ.

Without solving the cubic equation,

(a) determine the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(3)

(b) find a cubic equation that has roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$, giving your answer in the form $x^3 + ax^2 + bx + c = 0$, where a, b and c are integers to be determined.

(3)

(Total for question = 6 marks)

Q3.

The cubic equation

$$x^3 + 3x^2 - 8x + 6 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha - 1)$, $(\beta - 1)$ and $(\gamma - 1)$, giving your answer in the form $w^3 + pw^2 + qw + r = 0$, where p, q and r are integers to be found.

(5)

(Total for question = 5 marks)

Q4.

The cubic equation

$$2x^3 + 6x^2 - 3x + 12 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$, giving your answer in the form $pw^3 + qw^2 + rw + s = 0$, where p, q, r and s are integers to be found.

(Total for question = 5 marks)

Q5.

The cubic equation

$$9x^3 - 5x^2 + 4x + 7 = 0$$

has roots α, β and γ.

Without solving the equation, find the cubic equation whose roots are $(3\alpha - 2)$, $(3\beta - 2)$ and $(3\gamma - 2)$, giving your answer in the form $aw^3 + bw^2 + cw + d = 0$, where a, b, c and d are integers to be determined.

(Total for question = 5 marks)

Q6.

The roots of the quartic equation

$$3x^4 + 5x^3 - 7x + 6 = 0$$

are α, β, γ and δ

Making your method clear and without solving the equation, determine the exact value of

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

(3)

(ii) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} + \frac{2}{\delta}$

(3)

(iii) $(3 - \alpha)(3 - \beta)(3 - \gamma)(3 - \delta)$

(3)

(Total for question = 9 marks)

Mark Scheme – Roots of Polynomials

Q1.

Question	Scheme	Marks	AOs
	$w = 2z+1 \Rightarrow z = \frac{w-1}{2}$	B1	3.1a
	$\left(\frac{w-1}{2}\right)^3 - 3\left(\frac{w-1}{2}\right)^2 + \left(\frac{w-1}{2}\right) + 5 = 0$	M1	3.1a
	$\frac{1}{8}\left(w^3 - 3w^2 + 3w - 1\right) - \frac{3}{4}\left(w^2 - 2w + 1\right) + \frac{w-1}{2} + 5 = 0$		
	$w^3 - 9w^2 + 19w + 29 = 0$	M1 A1 A1	1.1b 1.1b 1.1b
		(5)	
ALT 1	$\alpha + \beta + \gamma = 3, \alpha\beta + \beta\gamma + \alpha\gamma = 1, \alpha\beta\gamma = -5$	B1	3.1a
	New sum $= 2(\alpha+\beta+\gamma)+3 = 9$	M1	3.1a
	New pair sum $= 4(\alpha\beta+\beta\gamma+\gamma\alpha)+4(\alpha+\beta+\gamma)+3 = 19$		
	New product $= 8\alpha\beta\gamma + 4(\alpha\beta+\beta\gamma+\gamma\alpha)+2(\alpha+\beta+\gamma)+1 = -29$		
	$w^3 - 9w^2 + 19w + 29 = 0$	M1 A1 A1	1.1b 1.1b 1.1b
		(5)	
			(5 marks)

Notes

B1: Selects the method of making a connection between z and w by writing $z = \frac{w-1}{2}$

M1: Applies the process of substituting their $z = \frac{w-1}{2}$ into $z^3 - 3z^2 + z + 5 = 0$

(Allow $z = 2w + 1$)

M1: Manipulates their equation into the form $w^3 + pw^2 + qw + r (= 0)$ having substituted their z in terms of w. Note that the "= 0" can be missing for this mark.

A1: At least two of p, q, r correct. Note that the "= 0" can be missing for this mark.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of w but the final answer must be in terms of w.

ALT1

B1: Selects the method of giving three correct equations containing α, β and γ

M1: Applies the process of finding the new sum, new pair sum, new product

M1: Applies $w^3 - (\text{new sum})w^2 + (\text{new pair sum})w - (\text{new product})(= 0)$ or identifies p as –(new sum) q as (new pair sum) and r as –(new product)

A1: At least two of p, q, r correct.

A1: Fully correct equation including "= 0"

The first 4 marks are available if another letter is used instead of w but the final answer must be in terms of w.

Q2.

Question	Scheme	Marks	AOs
(a)	$\alpha\beta\gamma = -\frac{1}{3}$ and $\alpha\beta+\alpha\gamma+\beta\gamma = -\frac{4}{3}$	B1	3.1a
	$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma} = \frac{-4/3}{-1/3}$	M1	1.1b
	$=4$	A1	1.1b
		(3)	
(b)	$\left\{\alpha+\beta+\gamma = -\frac{1}{3}\right\}$ New product $= \frac{1}{\alpha}\times\frac{1}{\beta}\times\frac{1}{\gamma} = \frac{1}{\alpha\beta\gamma} = \frac{1}{-1/3} = ...(-3)$ New pair sum $\frac{1}{\alpha\beta}+\frac{1}{\beta\gamma}+\frac{1}{\alpha\gamma} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{-1/3}{-1/3} = ...(1)$	M1	3.1a
	$x^3 - (\text{part (a)})x^2 + (\text{new pair sum})x - (\text{new product})(=0)$	M1	1.1b
	$x^3 - 4x^2 + x + 3 = 0$	A1	1.1b
		(3)	
	Alternative e.g. $z = \frac{1}{x} \Rightarrow \frac{3}{x^3}+\frac{1}{x^2}-\frac{4}{x}+1=0$	M1	3.1a
	$x^3 - 4x^2 + x + 3 = 0$	M1 A1	1.1b 1.1b
		(3)	
			(6 marks)

Notes:

(a)

B1: Correct values for the product and pair sum of the roots

M1: A complete method to find the sum of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. Must substitute in their values of the product and pair sum

A1: correct value 4

Note: If candidate does not divide by 3 so that $\alpha\beta\gamma = -1$ and $\alpha\beta+\alpha\gamma+\beta\gamma = -4$ the maximum they can score is B0 M1 A0

(b)

M1: A correct method to find the value of the new pair sum and the value of the new product

M1: Applies $x^3 - (\text{part (a)})x^2 + (\text{their new pair sum})x - (\text{their new product})(=0)$

A1: Fully correct equation, in any variable, including = 0

(b) Alternative

M1: Realises the connection between the roots and substitutes into the cubic equation

M1: Manipulates their equation into the form $x^3 + ax^2 + bx + c = 0$

A1: Fully correct equation in any variable, including = 0

Q3.

Question	Scheme	Marks	AOs
	$\{w = x - 1 \Rightarrow\}\ x = w + 1$	B1	3.1a
	$(w+1)^3 + 3(w+1)^2 - 8(w+1) + 6 = 0$	M1	3.1a
	$w^3 + 3w^2 + 3w + 1 + 3(w^2 + 2w + 1) - 8w - 8 + 6 = 0$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
ALT 1	$\alpha + \beta + \gamma = -3,\ \alpha\beta + \beta\gamma + \alpha\gamma = -8,\ \alpha\beta\gamma = -6$	B1	3.1a
	sum roots = $\alpha - 1 + \beta - 1 + \gamma - 1$	M1	3.1a
	$= \alpha + \beta + \gamma - 3 = -3 - 3 = -6$		
	pair sum = $(\alpha-1)(\beta-1) + (\alpha-1)(\gamma-1) + (\beta-1)(\gamma-1)$		
	$= \alpha\beta + \alpha\gamma + \beta\gamma - 2(\alpha+\beta+\gamma) + 3$		
	$= -8 - 2(-3) + 3 = 1$		
	product = $(\alpha - 1)(\beta - 1)(\gamma - 1)$		
	$= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha+\beta+\gamma) - 1$		
	$= -6 - (-8) - 3 - 1 = -2$		
	$w^3 + 6w^2 + w + 2 = 0$	M1	1.1b
		A1	1.1b
		A1	1.1b
		(5)	
		(5 marks)	

	Question Notes	
	B1	Selects the method of making a connection between x and w by writing $x = w + 1$
	M1	Applies the process of substituting their $x = w + 1$ into $x^3 + 3x^2 - 8x + 6 = 0$
	M1	Depends on previous M mark. Manipulating their equation into the form $w^3 + pw^2 + qw + r = 0$
	A1	At least two of p, q, r are correct.
	A1	Correct final equation.
ALT 1	B1	Selects the method of giving three correct equations each containing α, β and γ.
	M1	Applies the process of finding sum roots, pair sum and product.
	M1	Depends on previous M mark. Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - \text{their } \alpha\beta\gamma = 0$
	A1	At least two of p, q, r are correct.
	A1	Correct final equation.

Q4.

Question	Scheme	Marks	AOs
	$\{w = x+3 \Rightarrow\}\ x = w-3$	B1	3.1a
	$2(w-3)^3 + 6(w-3)^2 - 3(w-3) + 12\ (=0)$	M1	1.1b
	$2w^3 - 18w^2 + 54w - 54 + 6(w^2 - 6w + 9) - 3w + 9 + 12 (= 0)$		
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	M1 A1 A1	3.1a 1.1b 1.1b
		(5)	
ALT 1	$\alpha+\beta+\gamma = -\frac{6}{2} = -3,\ \alpha\beta+\beta\gamma+\alpha\gamma = -\frac{3}{2},\ \alpha\beta\gamma = -\frac{12}{2} = -6$	B1	3.1a
	sum roots $= \alpha+3+\beta+3+\gamma+3$ $= \alpha+\beta+\gamma+9 = -3+9 = 6$ pair sum $= (\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$ $= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha+\beta+\gamma) + 27$ $= -\frac{3}{2} + 6\times -3 + 27 = \frac{15}{2}$ product $= (\alpha+3)(\beta+3)(\gamma+3)$ $= \alpha\beta\gamma + 3(\alpha\beta+\alpha\gamma+\beta\gamma) + 9(\alpha+\beta+\gamma) + 27$ $= -6 + 3\times -\frac{3}{2} + 9\times -3 + 27 = -\frac{21}{2}$	M1	3.1a
	$w^3 - 6w^2 + \frac{15}{2}w - \left(-\frac{21}{2}\right)\ (=0)$	M1	1.1b
	$2w^3 - 12w^2 + 15w + 21 = 0$ (So $p = 2$, $q = -12$, $r = 15$ and $s = 21$)	A1 A1	1.1b 1.1b
		(5)	
		(5 marks)	

Notes		
	B1	Selects the method of making a connection between x and w by writing $x = w-3$
	M1	Applies the process of substituting their $x = aw \pm b$ into $2x^3 + 6x^2 - 3x + 12\ (=0)$ So accept e.g. if $x = \frac{w}{3}$ is used.
	M1	Depends on having attempted substituting either $x = w-3$ or $x = w+3$ into the equation. This mark is for manipulating their resulting equation into the form $pw^3 + qw^2 + rw + s(=0)$ ($p \neq 0$). The "= 0" may be implied for this.
See note	A1	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "= 0")
	A1	Correct final equation, including "=0". Accept integer multiples.
ALT 1	B1	Selects the method of giving three correct equations each containing α, β and γ.
	M1	Applies the process of finding sum roots, pair sum and product.
	M1	Applies $w^3 - (\text{their sum roots})w^2 + (\text{their pair sum})w - (\text{their product})\ (=0)$ Must be correct identities, but if quoted allow slips in substitution, but the "=0" may be implied.
See note	A1	At least three of p, q, r and s are correct in an equation with integer coefficients. (need not have "=0")
	A1	Correct final equation, including "=0". Accept multiples with integer coefficients.
Note: may use another variable than w for the first four marks, but the final equation must be in terms of w		
Notes: Do not isw the final two A marks – if subsequent division by 2 occurs then mark the final answer.		

Q5.

Question	Scheme	Marks	AOs
	$w = 3x - 2 \Rightarrow x = \dfrac{w+2}{3}$	B1	3.1a
	$9\left(\dfrac{w+2}{3}\right)^3 - 5\left(\dfrac{w+2}{3}\right)^2 + 4\left(\dfrac{w+2}{3}\right) + 7 = 0$	M1	3.1a
	$\dfrac{1}{3}\left(w^3 + 6w^2 + 12w + 8\right) - \dfrac{5}{9}\left(w^2 + 4w + 4\right) + \dfrac{4}{3}(w+2) + 7 = 0$		
	$3w^3 + 13w^2 + 28w + 91 = 0$	dM1 A1 A1	1.1b 1.1b 1.1b
		(5)	
	Alternative:		
	$\alpha + \beta + \gamma = \dfrac{5}{9}, \alpha\beta + \beta\gamma + \alpha\gamma = \dfrac{4}{9}, \alpha\beta\gamma = -\dfrac{7}{9}$	B1	3.1a
	New sum $= 3(\alpha + \beta + \gamma) - 6 = -\dfrac{13}{3}$	M1	3.1a
	New pair sum $= 9(\alpha\beta + \beta\gamma + \gamma\alpha) - 12(\alpha + \beta + \gamma) + 12 = \dfrac{28}{3}$		
	New product $= 27\alpha\beta\gamma - 18(\alpha\beta + \beta\gamma + \gamma\alpha) + 12(\alpha + \beta + \gamma) - 8 = -\dfrac{91}{3}$		
	$w^3 - \left(-\dfrac{13}{3}\right)w^2 + \dfrac{28}{3}w - \left(-\dfrac{91}{3}\right) = 0$	dM1	1.1b
	$3w^3 + 13w^2 + 28w + 91 = 0$	A1 A1	1.1b 1.1b
		(5)	
			(5 marks)

Notes
B1: Selects the method of making a connection between x and w by writing $x = \dfrac{w+2}{3}$ Condone the use of a different letter than w M1: Applies the process of substituting $x = \dfrac{w+2}{3}$ into $9x^3 - 5x^2 + 4x + 7 = 0$ dM1: Depends on the previous M mark. Manipulates their equation into the form $aw^3 + bw^2 + cw + d (= 0)$. Condone the use of a different letter then w consistent with B1 mark. A1: At least two of a, b, c, d correct A1: Fully correct equation, must be in terms of w **Alternative:** B1: Selects the method of giving three correct equations containing α, β and γ M1: Applies the process of finding the new sum, new pair sum, new product dM1: Depends on the previous M mark. Applies $w^3 - (\text{new sum})w^2 + (\text{new pair sum})w - (\text{new product})(= 0)$ condone the use of any letter here. A1: At least two of a, b, c, d correct A1: Fully correct equation in term of w

Q6.

Question	Scheme	Marks	AOs
(i)	$\sum \alpha_i = -\frac{5}{3}$ and $\sum \alpha_i\alpha_j = 0$ This mark can be awarded if seen in part (ii) or part (iii)	B1	3.1a
	So $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2\left(\sum \alpha_i\alpha_j\right) = \ldots$	M1	1.1b
	$= \frac{25}{9} - 2\times 0 = \frac{25}{9}$	A1	1.1b
		(3)	
(ii)	$\sum \alpha_i\alpha_j\alpha_k = \frac{7}{3}$ and $\prod \alpha_i = 2$ or for $x = \frac{2}{w}$ used in equation This mark can be awarded if seen in part (i) or part (iii)	B1	2.2a
	So $2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right) = 2\times\frac{\sum \alpha_i\alpha_j\alpha_k}{\alpha\beta\gamma\delta} = 2\times\frac{'\frac{7}{3}'}{'\frac{6}{3}'}$ or for $3\left(\frac{16}{w^4}\right)+5\left(\frac{8}{w^3}\right)-7\left(\frac{2}{w}\right)+6=0 \Rightarrow 6w^4-14w^3+\ldots=0$ leading to $\frac{14}{6}$	M1	1.1b
	$\left(=2\times\frac{7/3}{2}\right)\left(=\frac{14}{6}\right)=\frac{7}{3}$	A1	1.1b
		(3)	
(iii)	$(3-\alpha)(3-\beta)(3-\gamma)(3-\delta) = \ldots$ expands all four brackets Or equation with these roots is $3(3-x)^4+5(3-x)^3-7(3-x)+6=0$	M1	3.1a
	$= 81-27\left(\sum \alpha_i\right)+9\left(\sum \alpha_i\alpha_j\right)-3\left(\sum \alpha_i\alpha_j\alpha_k\right)+\prod \alpha_i$ $= 81-27\left(-\frac{5}{3}\right)+9(0)-3\left(\frac{7}{3}\right)+2$ Or expands to fourth power and constant terms and attempts product of roots $3x^4+\ldots+3\times 3^4+5\times 3^3-7\times 3+6 \to \prod \alpha_i = \frac{"363"}{3}$	dM1	1.1b
	$=121$	A1	1.1b
		(3)	
		(9 marks)	

Notes:

(i)

B1: Correct sum and pair sum of roots seen or implied. Must realise the pair sum is zero.

Note: These values can be seen anywhere in the candidate's solution

M1: Uses correct expression for the sum of squares.

A1: $\frac{25}{9}$. Allow this mark from incorrect sign on sum of squares (but they will score B0 if the sign is incorrect).

(ii)

B1: Correct triple sum and product of roots seen or implied. May be stated in (i). Alternatively, this may be scored for sight of $x = \frac{2}{w}$ used as a transformation in the equation.

Note: These values can be seen anywhere in the candidate's solution

M1: Substitutes their values into $2 \times \frac{\sum \alpha_i \alpha_j \alpha_k}{\alpha\beta\gamma\delta} = \ldots$ In the alternative it is for rearranging the equation to a quartic in w and uses to find the sum of the roots.

A1: $\frac{7}{3}$ Allow this mark from incorrect sign of both triple sum and product (but they will score B0 if the sign is incorrect).

(iii)

M1: A correct method to find the value used – may recognise structure as scheme, may expand the expression in stages, or may attempt to use a linear transformation $(3 - x)$ or e.g. $(3 - w)$ in original equation. Condone slips as long as the intention is clear.

dM1: Dependent on previous method mark. Uses at least 2 values of their sum of roots etc. in their expression. If using a linear shift this is for expanding to find the coefficient of x^4 and constant term and attempts product of roots by dividing the constant term by the coefficient of x^4.

A1: 121.