Proof by induction Cheat Sheet

Edexcel Core Pure 1

Proof by mathematical induction is a method used to prove statements that involve positive integers.

You need to be able to use proof by induction to prove results that involve:

- Summation of series
- Divisibility statements
- Matrices

We will discuss each type of question separately.

$\mathbb{Z}^+$ is used to denote the set of all positive integers.

The general method

When proving a statement by induction, there are four steps you must follow:

1. **Basis:** Prove the statement is true for a starting value (usually $n = 1$).
2. **Assumption:** Assume the statement is true for $n = k$, where k is a positive integer.
3. **Inductive:** Use the assumption to prove that the statement is true for $n = k + 1$.
4. **Conclusion:** Write a conclusion that verifies the statement is true for all positive integers, n.

The inductive step usually requires the most work, and therefore is where most of the marks will come from.

Series

- When proving results involving series, it is useful to write down what you need to prove in the inductive step before starting.
- During the inductive step, you will need to use the fact that $\sum_{r=1}^{k+1} f(r) = \sum_{r=1}^{k} f(r) + f(k+1)$.

Example 1: Prove by induction that, for $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n}(4r^3 - 3r^2 + r) = n^3(n+1)$$

Start by making a note of what you want to prove in the inductive step.	$\sum_{r=1}^{k+1}(4r^3 - 3r^2 + r) = (k+1)^3(k+2)$
We start with the basis step; we show the $LHS = RHS$:	For $n = 1$: $LHS = 4(1)^3 - 3(1)^2 + 1 = 2$ $RHS = 1^3(1+1) = 2 = LHS$ ∴ the statement is true for $n = 1$.
Next we carry out the assumption step:	Assume that the statement is true for $n = k$. i.e. $\sum_{r=1}^{k}(4r^3 - 3r^2 + r) = k^3(k+1)$
Now for the inductive step. We need to prove that the statement is true for $n = k + 1$. Use the fact stated in the second bullet point above.	$\sum_{r=1}^{k+1}(4r^3 - 3r^2 + r) = \sum_{r=1}^{k}(4r^3 - 3r^2 + r) + (k+1)^{th}\ term$ $\sum_{r=1}^{k+1}(4r^3 - 3r^2 + r) = k^3(k+1) + 4(k+1)^3 - 3(k+1)^2 + k + 1$
Simplifying by factoring out $(k+1)$:	$= (k+1)[k^3 + 4(k+1)^2 - 3(k+1) + 1]$ $= (k+1)(k^3 + 4k^2 + 5k + 2)$
Looking back at what we want to show, we can notice that the cubic we have factorises to $(k+1)^2(k+2)$. This gives the required result.	$= (k+1)(k+1)^2(k+2)$ $= (k+1)^3(k+2)$ ∴ the statement is true for $n = k + 1$.
Finish by writing the conclusion.	So we have proven the statement true for $n = 1$. When we assumed it to be true for $n = k$, we showed that it was also true for $n = k + 1$. ∴ by mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.

This means that the statement is true for all positive integers, n.

Divisibility statements

- During the inductive step, it is useful to consider $f(k+1) - cf(k)$, where c is a constant chosen in order to cancel out terms.

Example 2: Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 8^n - 3^n \text{ is divisible by } 5$$

We start with the basis step; we show the $LHS = RHS$ for $n = 1$:	For $n = 1$: $f(1) = 8^1 - 3^1 = 5 = 5(1)$ So the statement is true for $n = 1$.
Next we carry out the assumption step:	Assume that the statement is true for $n = k$. i.e. $8^k - 3^k$ is divisible by 5.
Now for the inductive step. We need to prove that the statement is true for $n = k + 1$. We consider $f(k+1) - 3f(k)$, since this causes the terms in 3^k to cancel out. We could also consider $f(k+1) - 8f(k)$ so that the terms in 8^k would cancel out.	$f(k+1) = 8^{k+1} - 3^{k+1} = 8(8^k) - 3(3^k)$ $f(k+1) - 3f(k) = 8(8^k) - 3(3^k) - 3(8^k) + 3(3^k)$ $f(k+1) - 3f(k) = 5(8^k)$
Making $f(k+1)$ the subject:	$f(k+1) - 3f(k) = 5(8^k)$ $f(k+1) = 3f(k) + 5(8^k)$
In general, if two terms are divisible by k, then their sum will also be divisible by k.	$f(k)$ we assumed to be divisible by 5 and $5(8^k)$ is clearly divisible by 5. So the sum of these terms will also be divisible by 5. Hence the statement is true for $n = k + 1$.
Finish by writing the conclusion.	So we have proven the statement true for $n = 1$. When we assumed it to be true for $n = k$, we showed that it was also true for $n = k + 1$. ∴ by mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.

Example 3: Prove by induction that, for $n \in \mathbb{Z}^+$

$$f(n) = 3^{3n-2} + 2^{3n+1} \text{ is divisible by } 19$$

We start with the basis step; we show the $LHS = RHS$ for $n = 1$:	For $n = 1$: $f(1) = 3^{3-2} + 2^{3+1} = 3 + 16 = 19 = 19(1)$ So the statement is true for $n = 1$.
Next we carry out the assumption step:	Assume that the statement is true for $n = k$. i.e. $3^{3k-2} + 2^{3k+1}$ is divisible by 19
Now for the inductive step. We need to prove that the statement is true for $n = k + 1$.	$f(k+1) = 3^{3(k+1)-2} + 2^{3(k+1)+1} = 3^{3k+1} + 2^{3k+4}$
Now we manipulate the powers to match those of $f(k)$. We do this so that terms will cancel out in the next step.	$f(k+1) = 3^{3k-2+3} + 2^{3k+1+3}$ $= 3^3(3^{3k-2}) + 2^3(2^{3k+1})$ $= 27(3^{3k-2}) + 8(2^{3k+1})$
We consider $f(k+1) - 8f(k)$, since this causes the terms in 2^{3k+1} to cancel out. We could also consider $f(k+1) - 27f(k)$ so that the terms in 3^{3k-2} would cancel out.	$f(k+1) - 8f(k) = 27(3^{3k-2}) + 8(2^{3k+1}) - 8(3^{3k-2}) - 8(2^{3k+1})$ $f(k+1) - 8f(k) = 19(3^{3k-2})$
Making $f(k+1)$ the subject:	$f(k+1) = 8f(k) + 19(3^{3k-2})$
In general, if two terms are divisible by k, then their sum will also be divisible by k.	$f(k)$ we assumed to be divisible by 19 and $19(3^{3k-2})$ is clearly divisible by 19. So the sum of these terms will also be divisible by 19. Hence the statement is true for $n=k+1$.
Finish by writing the conclusion.	So we have proven the statement true for $n = 1$. When we assumed it to be true for $n = k$, we showed that it was also true for $n = k + 1$. ∴ by mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.

Matrices

- When proving results involving matrices, it is useful to write down what you need to prove in the inductive step before starting.
- During the inductive step, you will need to use the fact that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{k+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{k} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Example 4: Prove by induction that, for $n \in \mathbb{Z}^+$ $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{n} = \begin{pmatrix} 1 & 1-2^n \\ 0 & 2^n \end{pmatrix}$

Start by making a note of what you want to prove in the inductive step.	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1-2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}$
We start with the basis step; we show the $LHS = RHS$ for $n = 1$:	For $n = 1$: $LHS = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{1} = RHS = \begin{pmatrix} 1 & 1-2 \\ 0 & 2 \end{pmatrix}$ ∴ true for $n = 1$.
Next we carry out the assumption step:	Assume that the statement is true for $n = k$. i.e. $\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k} = \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix}$
Now for the inductive step. We need to prove that the statement is true for $n = k + 1$. Using the above bullet point:	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$
Using our assumption step and multiplying the matrices out:	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1-2^k \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1+2(1-2^k) \\ 0 & 2(2^k) \end{pmatrix}$
Simplifying the entries:	$= \begin{pmatrix} 1 & 1-2(2^k) \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1-(2^{k+1}) \\ 0 & 2^{k+1} \end{pmatrix}$ as required.
Finishing by writing the conclusion.	So we have proven the statement true for $n = 1$. When we assumed it to be true for $n = k$, we showed that it was also true for $n = k + 1$. ∴ by mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.

Example 5: Prove by induction that, for $n \in \mathbb{Z}^+$ $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{n} = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

Start by making a note of what you want to prove in the inductive step.	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$
We start with the basis step; we show the $LHS = RHS$ for $n = 1$:	For $n = 1$: $LHS = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{1}$ $RHS = \begin{pmatrix} 2(1)+1 & -2(1) \\ 2(1) & 1-2(1) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = LHS$ ∴ true for $n = 1$.
Next we carry out the assumption step:	Assume that the statement is true for $n = k$. i.e. $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}$
Now for the inductive step. We need to prove that the statement is true for $n = k + 1$. Using the above bullet point:	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$
Using our assumption step:	$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$
Multiplying the matrices out:	$\begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ $= \begin{pmatrix} 3(2k+1)-4k & -2(2k+1)+2k \\ 3(2k)+2(1-2k) & -2(2k)-1(1-2k) \end{pmatrix}$
Simplifying	$= \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$
Rewriting to show clearly that we have achieved the desired result with $n = k + 1$:	$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$ ∴ true for $n = k + 1$
Finishing by writing the conclusion.	So we have proven the statement true for $n = 1$. When we assumed it to be true for $n = k$, we showed that it was also true for $n = k + 1$. ∴ by mathematical induction the statement is true for all $n \in \mathbb{Z}^+$.