Linear transformations Cheat Sheet

A linear transformation describes how a general point is transformed. The new point is called the image.

Properties

A linear transformation only involves linear terms in x and y. Below are three different transformations. The only one that is a **linear** transformation is T since the transformation matrix has all entries written in the form $ax + by$.

$$S: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ y-8 \end{pmatrix} \qquad T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 4x-y \\ x+2y \end{pmatrix} \qquad U: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 4y \\ -x^2 \end{pmatrix}$$

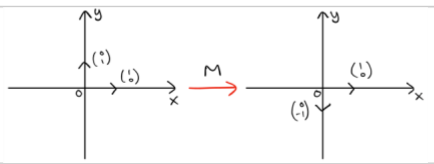
- Any linear transformation can be represented by a matrix
- Linear transformations always map the origin onto itself.
- The linear transformation $T: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ can be represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Any linear transformation can be defined by its effect on the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. You can use the following fact to find the transformation that a given matrix represents.

- If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$.

Example 1: $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Describe fully the transformation represented by M.

We find what happens to the unit vectors under M:	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
Drawing a diagram to visualise the effect M had on the unit vectors:	
The unit vectors have been reflected in the $x - axis$.	(1,0) is unchanged by M while $(0, 1)$ is transformed to $(0, -1)$. We can conclude that M represents a reflection in the x −axis.

Reflections, rotations, enlargements and stretches

- Points which are mapped to themselves under a given transformation are known as invariant points.
- Lines which are mapped to themselves under a given transformation are known as invariant lines.

The below table details the different matrices that correspond to particular transformations you need to be familiar with, as well as their respective invariant points and lines. You will be given the rotation matrix in the formula booklet, as well as the matrix for a reflection in the line $y = (tan\theta)x$.

Transformation	Matrix	Invariant points	Invariant lines
Reflection in the y −axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	All points on the y −axis	$x = 0$ and $y = k$ for any k
Reflection in the x −axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	All points on the x −axis	$y = 0$ and $x = k$ for any k
Reflection in the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	All points on the line $y = x$	The lines $y = -x$ and $y = -x + k$ for any k
Reflection in the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	All points on the line $y = -x$	The lines $y = -x$ and $y = x + k$ for any k
Rotation through angle θ anticlockwise about the origin	$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$	Only the origin $(0, 0)$	For $\theta \neq 180°$ there are no invariant lines. For $\theta = 180°$ any line passing through the origin is an invariant line.
Stretch of scale factor a parallel to the x −axis and a stretch of scale factor b parallel to the y −axis **If $a = b$, the transformation is an enlargement with scale factor a**	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	Only the origin $(0, 0)$	The x and y axes.
Stretch of scale factor a parallel to the x −axis only	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$	All points on the y −axis	Any line parallel to the x −axis
Stretch of scale factor b parallel to the y −axis only	$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$	All points on the x −axis	Any line parallel to the y −axis

Remember that you can always use the method from example 1 to find the transformation represented by a particular matrix if you are unsure of what it represents.

The determinant

If the determinant of M is negative then the shape has been reflected.

- If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents a linear transformation, then $detM$ represents the scale factor for the change in area.

The following formula is useful for questions involving the change in area as a result of a linear transformation.

- $Area_{image} = Area_{object} \times |detM|$

$Area_{object}$ represents the area before the transformation is applied.

Example 2: $M = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

(a) Describe fully the transformation represented by M.

A 2D shape with area k is transformed using the transformation represented by M.

(b) Given that the image of the shape has area 24, find the value of k.

Identifying the correct transformation:	This is a stretch of scale factor 2 parallel to the x −axis and a stretch of scale factor −3 parallel to the y −axis.
Using $Area_{image} = Area_{object} \times \lvert detM \rvert$	$24 = k \times \lvert detM \rvert$
Finding the determinant and substituting back into the equation formed in the previous step:	$detM = 2(-3) - 0 = -6$ $\therefore 24 = k \times \lvert -6 \rvert \Rightarrow k = \frac{24}{6} = 4$

Successive transformations

- The matrix $\boldsymbol{PQ}$ represents the transformation Q followed by the transformation P, where these transformations are represented by matrices $\boldsymbol{Q}$ and $\boldsymbol{P}$ respectively.

Example 3: Use matrices to show that a reflection in the y −axis followed by a reflection in the line $y = -x$ is equivalent to a rotation of 90° anticlockwise about $(0, 0)$.

Identifying the correct matrices for the two reflections:	Let U represent a reflection in the y −axis and V represent a reflection in the line $y = -x$. Then $\boldsymbol{U} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{V} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
The given combination of transformations will be given by VU	$VU = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
Substituting $\theta = 90°$ into the rotation matrix and showing it to be equal to VU:	Anticlockwise rotation of 90° about the origin is given by $\begin{pmatrix} cos90° & -sin90° \\ sin90° & cos90° \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = VU$ as required.

Linear transformations in three dimensions

Just like in two dimensions, a linear transformation in three dimensions can be defined by its effect on the unit vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. You need to be familiar with the following linear transformations in three dimensions:

Transformation	Matrix
Reflection in the plane $x = 0$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Reflection in the plane $y = 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Reflection in the plane $z = 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
Rotation anticlockwise, angle θ, about the x −axis	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$
Rotation anticlockwise, angle θ, about the y −axis	$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$
Rotation anticlockwise, angle θ, about the z −axis	$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The inverse of a linear transformation

- If $\boldsymbol{A}$ is a matrix representing a transformation, then the matrix $\boldsymbol{A}^{-1}$ has the effect of reversing the transformation described by $\boldsymbol{A}$.

Example 4: $B = \begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix}$

The matrix B represents a linear transformation, T, which maps coordinates (x,y) to position (8,12). Find the coordinates (x,y).

Find the inverse matrix B^{-1}	$B^{-1} = \frac{1}{\det B}\begin{pmatrix} -5 & -4 \\ 2 & 2 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -5 & -4 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2.5 & 2 \\ -1 & -1 \end{pmatrix}$
Form an equation to find (x,y)	$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = B^{-1}\begin{pmatrix} 8 \\ 12 \end{pmatrix}$
Perform calculations to find (x,y)	$\begin{pmatrix} x \\ y \end{pmatrix} = B^{-1}\begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 2.5 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 44 \\ -20 \end{pmatrix}$

Finding invariant points and lines

Some questions will require you to find the points and/or lines that are invariant under a transformation that does not represent a reflection, rotation, enlargement or stretch.

- To find the invariant points under a transformation represented by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, you need to solve the system $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$ for X and Y.
- To find the invariant lines under a transformation represented by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, you need to solve the system $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ MX + C \end{pmatrix} = \begin{pmatrix} X' \\ MX' + C \end{pmatrix}$ for M and C.

Example 5: $P = \begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}$

The matrix P represents a linear transformation, T, of the plane.

(a) Describe the invariant points of the transformation T.

(b) Describe the invariant lines of the transformation T.

(a) This is the system we need to solve:	$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$ $\Rightarrow \begin{pmatrix} 3X + 3Y \\ 4X + 7Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$
Equating the two rows:	$3X + 3Y = X$ [1] $4X + 7Y = Y$ [2]
Simplifying the first equation, making Y the subject:	$2X + 3Y = 0 \Rightarrow Y = -\frac{2}{3}X$
Simplifying the second equation, making Y the subject:	$4X + 6Y = 0 \Rightarrow Y = -\frac{4}{6}X = -\frac{2}{3}X$
Both equations yield the same result, so we can confirm we have the correct conclusion:	Both equations are consistent; the invariant points lie on the line $Y = -\frac{2}{3}X$.
(b) This is the system we need to solve:	$\begin{pmatrix} 3 & 3 \\ 4 & 7 \end{pmatrix}\begin{pmatrix} X \\ MX + C \end{pmatrix} = \begin{pmatrix} X' \\ MX' + C \end{pmatrix}$ $\begin{pmatrix} 3X + 3(MX + C) \\ 4X + 7(MX + C) \end{pmatrix} = \begin{pmatrix} X' \\ MX' + C \end{pmatrix}$
Equating both rows:	$3X + 3(MX + C) = X'$ [1] $4X + 7(MX + C) = MX' + C$ [2]
Taking [1] and substituting into [2]:	$4X + 7(MX + C) = M(3X + 3(MX + C) + C$
Simplifying the resultant equation:	$4X + 4MX + 6C = 3M^2X + 3MC$
Factoring out the X:	$X(-3M^2 + 4M + 4) + 3C(2 - M) = 0$
Factorising the quadratic in M:	$X(2 - M)(3M + 2) + 3C(2 - M) = 0$
Now we need to think: what values could M and C take so that this equation is satisfied? i.e. what could M and C be so that the $LHS = RHS$?	$M = 2 \Rightarrow C$ could be anything. So $Y = 2X + C$ is an invariant line for any C.
Notice that another solution is given by $M = -2/3$ and $C = 0$.	$M = -2/3$ AND $C = 0$ would also satisfy the equation. So $Y = -\frac{2}{3}X$ is also an invariant line.