Matrices Cheat Sheet

Edexcel Core Pure 1

The aim of this chapter is to introduce the new concept of matrices, a mathematical object that can concisely store a lot of information to solve problems. We will look at different types of matrices and the operations we can perform on them. While a lot of the techniques covered in this topic seem arbitrary, matrices are surprising useful in abstracting problems and will be used extensively in future chapters for linear transformations and even inductive proofs.

What are matrices?

Matrices store numbers, called elements, in a grid like structure, and these numbers usually represent quantities in real world problems. Like vectors, matrices are usually denoted using a bold face letter, capitalised to show it's a matrix.

Example: A few matrices

A few 2×2 matrices	$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1.5 & 1/3 \\ -9 & \pi \end{pmatrix}$
Matrices of other sizes. Matrices can hold any kind of number and even variables.	$\mathbf{C} = \begin{pmatrix} 0 & 1 & \sin\pi \\ \log 2 & y & 6 \\ e^x & 1+i & 3 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 0 & 1 & 4 & 5 \\ 0 & 2 & 0 & 2 \end{pmatrix}$

Each of the above is called a matrix. When the number of rows is equal to number of columns, it is called a square matrix. Here, all except matrix **D** are square matrices. The dimensions of a matrix is represented by $m \times n$, where m is the number of rows and n is the number of columns. For the above 4 matrices, the dimensions are 2×2, 2×2, 3×3 and 2×4.

Operations with matrices

- Addition and Subtraction

  Matrices can be added or subtracted if they have the same dimensions. If two matrices have the same dimensions, they are said to be additively conformable. Addition and subtraction is performed in an element-wise manner

  Example: Add the matrices $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 5 & -9 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 & 7 \\ -4 & -8 \end{pmatrix}$.

Write out the sum	$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 3 \\ 5 & -9 \end{pmatrix} + \begin{pmatrix} 6 & 7 \\ -4 & -8 \end{pmatrix}$
Add the elements in corresponding positions of the two matrices.	$\begin{pmatrix} 1 & 3 \\ 5 & -9 \end{pmatrix} + \begin{pmatrix} 6 & 7 \\ -4 & -8 \end{pmatrix} = \begin{pmatrix} 1+6 & 3+7 \\ 5+-4 & -9+-8 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 1 & -17 \end{pmatrix}$

- Scalar Multiplication

  Any matrix can be multiplied by a scalar. Each element in the matrix is multiplied by the same scalar. Scalars *scale* matrices and vectors.

  Example: Multiply the matrix $\mathbf{C} = \begin{pmatrix} 0 & -2 & 0 \\ 5 & 1 & 3 \end{pmatrix}$ by the scalar $k = -3$.

Set up the scalar multiplication	$k\mathbf{C} = k\begin{pmatrix} 0 & -2 & 0 \\ 5 & 1 & 3 \end{pmatrix} = -3\begin{pmatrix} 0 & -2 & 0 \\ 5 & 1 & 3 \end{pmatrix}$
Perform scalar multiplication by multiplying each element by the scalar.	$-3\begin{pmatrix} 0 & -2 & 0 \\ 5 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 0\times-3 & -2\times-3 & 0\times-3 \\ 5\times-3 & 1\times-3 & 3\times-3 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 0 \\ -15 & -3 & -9 \end{pmatrix}$

- Matrix Multiplication

  Matrix multiplication is very different to any multiplication you met before. Each row of the first matrix is multiplied by columns of the second matrix in an element wise manner, and then summed to give an element of the product matrix.

  Example: Multiply the matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$.

Check the dimensions of the matrices allow for multiplication.	The matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2 matrix. The matrix $B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ is also a 2×2 matrix. So, since the number of columns of A equals the number of rows of B, the inner dimensions are equal and matrix multiplication can be carried out.
Perform matrix multiplication.	$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1\times5+2\times7 & 1\times6+2\times8 \\ 3\times5+4\times7 & 3\times6+4\times8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$

Your calculators can perform matrix multiplication if you need to check your working.
Two matrices **A** and **B** can only be multiplied if the number of columns of **A** is equal to the number of rows of **B**. Such matrices are multiplicatively conformable, i.e. if **A** has dimensions $n \times m$ and **B** has dimensions $p \times q$, **AB** exists if $m = p$ and it will have dimensions $n \times q$.

Example: Multiply the matrices $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 5 & 6 & 3 & 2 \\ 7 & 8 & 2 & 1 \\ 6 & 0 & 2 & 0 \end{pmatrix}$

Perform matrix multiplication on two differently shaped but multiplicatively conformable matrices	$\mathbf{AB} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} 5 & 6 & 3 & 2 \\ 7 & 8 & 2 & 1 \\ 6 & 0 & 2 & 0 \end{pmatrix}$
The product of the 2×3 and 3×4 matrix is a 2×4 matrix, as expected.	$= \begin{pmatrix} 37 & 22 & 13 & 4 \\ 73 & 50 & 27 & 10 \end{pmatrix}$

The order of matrix multiplication matters. In general, **AB** ≠ **BA** Furthermore, when dealing with non-square matrices **AB** may exist while **BA** may not.

Special Matrices

Zero Matrix

The zero matrix is a matrix where all entries are 0. Multiplying any conformable matrix with the zero matrix always produces the zero matrix.

The Identity Matrix

The identity matrix is a square matrix where all the diagonal entries are 1, and all the other entries are 0. Multiplying any conformable matrix with the identity matrix returns the original matrix.

A 3×3 zero matrix.	$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	The 2×2 and 3×3 identity matrices.	$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Determinant

The determinant is a function that takes a square matrix as an input and outputs a number. It can be represented using det() or vertical lines around the matrix.

For any 2x2 matrix:

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ab - cd}$$

For any 3x3 matrix, it is slightly more involved. The determinant can be found by expanding along any row or column. To expand along the first row, for each element in the first row you will need to calculate the determinants of the 2x2 matrix left over after deleting the row and column that element lives in (this is called the minor of that element).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

So, the determinant can be found by expanding along the first row:

$$\boxed{\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}}$$

Notice how the 2x2 minor matrices have alternating signs. Each minor is assigned a sign. Signed minors are called cofactors. The sign that goes with each cofactor alternates across the elements of the matrix:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

So, if we wanted to calculate the determinant by expanding along the second column for example, we will get the same number, calculated as:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

Example: Evaluate the determinant of $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 2 \\ 1 & -3 & 1 \end{pmatrix}$

Expand along any row or column. In this example, we use the first row.	$\det\begin{pmatrix} 1 & 1 & 0 \\ 2 & 4 & 2 \\ 1 & -3 & 1 \end{pmatrix} = 1\begin{vmatrix} 4 & 2 \\ -3 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix}$
Evaluate each of the 2×2 determinants	$= 1(4+6) - 1(2-2) + 0(-6-4) = 10$

If a matrix is found to have a determinant of 0, it is called a singular matrix. Otherwise, it is a non-singular matrix.

Inverting Matrices

Multiplying an nxn matrix by an n-dimensional vector produces another n-dimensional vector as an output (by following the rules of matrix multiplication). The input vector can be thought to be transformed by the matrix.

Example: Transform $\binom{2}{7}$ by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

They are multiplicatively conformable, perform matrix multiplication.	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0\times2+1\times7 \\ 1\times2+0\times7 \end{pmatrix}$
The product is another 2D vector	$= \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

The inverse of a matrix is a matrix that can transform the output vector back to its input vector. The idea of transforming vectors with matrices is covered in depth in the next chapter, but we'll look at how to compute the inverse of a matrix. In general,

$$\boxed{\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{C}^T}$$

Where $\mathbf{A}^{-1}$ is the inverse of **A**, and $\mathbf{C}^T$ is the transpose of the matrix of cofactors for each element.

The transpose operation swaps the rows and columns of a matrix (which results in a change of dimensions of non-square matrices):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

The cofactor matrix of a given square matrix is the square matrix where the elements are replaced by their cofactors:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \quad \mathbf{C} = \begin{pmatrix} +|d| & -|c| \\ -|b| & +|a| \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \quad \mathbf{C} = \begin{pmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

Therefore, using these ideas of transpositions and cofactors, we can work out the inverse of any 2x2 matrix to be:

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Writing out the general equation for any 3x3 matrix is too tedious to read, but it follows the same idea of finding the determinant and the transposed cofactor matrix.

Notice the scalar factor of 1/det**A** . This means singular matrices have no inverses, since their determinants are always 0. For this reason, non-singular matrices are often referred to as invertible matrices.

Since the inverse matrix maps the output vector back to the input vector, if we multiply a vector by a matrix and then its inverse, we should get back the vector we started with. Mathematically,

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

where **I** is the identity matrix, and **A** is any invertible square matrix.

Systems of Linear Equations

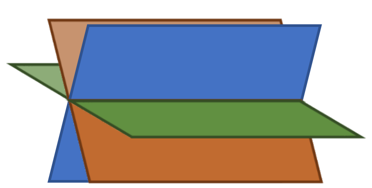
Matrices can compactly express sets of simultaneous linear equations for a large number of variables.

Example: Solve the following system of linear equations:

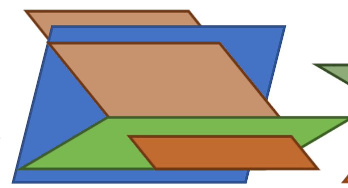
$$\begin{aligned} x - y + z &= 0 \\ 2x + y - 3z &= 1 \\ 2x + 2y + z &= 7 \end{aligned}$$

By expressing a system of linear equations in matrix form, they can quickly be solved by matrix inversion, rather than tedious substitution or elimination. You must have the same number of unique linear equations as there are variables to be able to have a solution.	$\begin{aligned} x - y + z &= 0 \\ 2x + y - 3z &= 1 \\ 2x + 2y + z &= 7 \end{aligned} \Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix}$
Check if the 3×3 matrix is invertible (non-singular). In this case the inverse exists.	$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$ $= (1+6) + (2+6) + (4-2) = 17$
Left-multiply both sides by the inverse of the 3×3 matrix. This makes the 3×3 matrix disappear on the LHS (as we are left with the identity), and x, y and z can be determined through matrix multiplication.	$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 2 & 2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

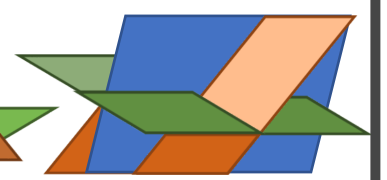
Note that for this method to work and give you the exact solution point formed by the intersection of the 3 planes, the matrix must be invertible. If the matrix is singular, there are two possible scenarios: there are zero solutions, or infinitely many solutions. These equations (for a 3x3 system) can be interpreted geometrically as:



Singular matrix with **infinitely many** solutions. The planes intersect on a line to form a sheaf.



Singular matrix with **zero** solutions. There is nowhere where all 3 planes meet. They form a prism.



Invertible matrices have **one** solution. The three planes intersect at a single point. (as in example above)