Volumes of Revolution Cheat Sheet

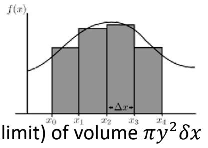
Edexcel Core Pure 1

In this chapter, you will learn how integration can be extended to calculate volumes as well as areas. The area under a curve is rotated about an axis to give a 3D shape. To understand this topic, you will need to be confident in the methods of integration you have previously encountered.

Volumes of revolution around the x-axis

When you use integration to find the area under a curve, you effectively split the area into infinitely many strips of infinitesimal width and sum up the area of each:

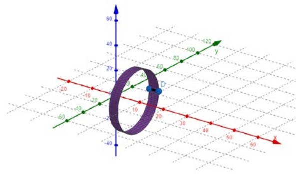
- $\delta A = y\delta x$
- $\lim\limits_{\delta x \to 0} \sum y\delta x = \int y dx$



If we then rotate this area 360 degrees around the x-axis, each strip forms a cylinder (in the limit) of volume $\pi y^2 \delta x$ (radius y and height δx). Hence, we conclude that the volume of revolution, V, formed when a function $y = f(x)$ is rotated through 2π (360°) about the x-axis over the range $x = [a, b]$ is given by:

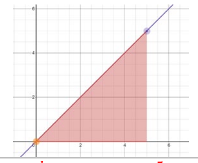
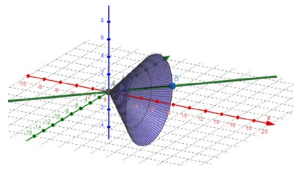
- $V = \pi \int_a^b y^2 \ dx$

A strip of finite length rotated about the x-axis, forming a cylinder.



To understand this, let's generate a shape for which we already know the volume.

Example 1: The finite region R represents the area bounded by the curve $y = x$, the x-axis and the line $x = 5$. The region R is rotated through 2π radians about the x-axis. Using integration, find the exact volume of the solid generated. Compare this to the value acquired using the classic geometric formula.

Start by sketching out the problem to be sure what region the question is referring to. This is a good habit to form early as it will help when the curves get more complex. The curve $y = x$ intersects the x-axis at $x = 0$ – we now have our limits: $a=0$, $b=5$.	
We use the above formula:	$V = \pi \int_a^b y^2 \ dx = \pi \int_0^5 x^2 \ dx$
Now we evaluate the integral, using the methods outlined in your Year 1 Maths course (don't forget the π!).	$\pi \int_0^5 x^2 \ dx = \pi \left[\frac{1}{3}x^3\right]_0^5 = \frac{125}{3}\pi$
If we rotate the curve, we get a cone of height $h = 5$, radius $r = 5$. We recall the formula for the volume of a cone.	$V = \frac{\pi r^2 h}{3}$ 
We evaluate V again using the volume of a cone formula.	$V = \frac{\pi r^2 h}{3} = \pi \times 5^2 \times \frac{5}{3} = \frac{125}{3}\pi$
Compare answers.	The value of V we get from the two methods matches, so we can conclude that our understanding of volumes of revolution is correct.

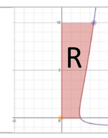
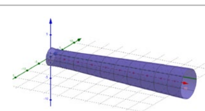
Volumes of revolution around the y-axis

We can use the same logic as above to derive the volume of a curve rotated about the y-axis (this is the same as just swapping the labels around).

- The volume of revolution, V, formed when a function $y = f(x)$ is rotated through 2π radians about the y-axis between the lines $y = a$ and $y = b$ is given by $V = \pi \int_a^b x^2 \ dy$

Always check you have your equation in the right form – i.e., x in terms of y – before you start to integrate. Some rearranging may be necessary.

Example 2: The diagram shows the curve with equation $x = \sqrt{y+1} + \frac{1}{y+1}$. The finite region R, shown in the diagram, is bounded by the curve, the x-axis, the y-axis and the line $y = 10$. Region R is rotated by 2π radians about the y-axis. Find the volume of the solid generated by this rotation.

The question gives us the region and a sketch, so we don't need to make our own. Double check which axis they are asking for before starting.	  Resultant volume of revolution
The curve is given in the correct form, $x = f(y)$, so all we need to do is square it before we begin integrating.	$x^2 = y + 1 + \frac{1}{(y+1)^2} + \frac{2}{\sqrt{y+1}}$
We use the formula for the rotation about the y-axis. To compute this integral, we can use substitution.	$u = y + 1, \frac{du}{dy} = 1$ $V = \pi \int_{u=1}^{11} u + \frac{1}{u^2} + \frac{2}{\sqrt{u}} \ du = \pi[\frac{1}{2}u^2 - \frac{1}{u} + 4\sqrt{u}]_1^{11} = \pi\left(\frac{121}{2} - \frac{1}{11} + 4\sqrt{11} - \frac{1}{2} + 1 - 4\right) = \left(4\sqrt{11} + \frac{626}{11}\right)\pi$

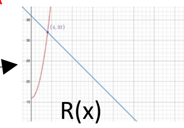
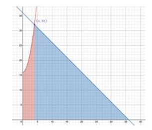
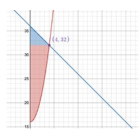
Adding and subtracting volumes

You can be asked to find a volume made from intersecting curves. In this case, you need to split the region up into areas of well-defined functions (i.e., that you can integrate) before applying the formulae we've learnt. If the question asks for a revolution about the x-axis, the line dividing your areas up should be parallel to the y-axis; for rotations about y, the division line should be parallel to the x-axis. Once you have the volume for each area, you can add or subtract them accordingly to give the final answer.

Example 3: The curves $f(x) = x^2 + 16$ and $g(x) = 36 - x$ for $x > 0$ intersect at a point P. Verify that P has the coordinates (4,32) and show how you would split the region for a revolution about

- the x-axis, where R is bounded by these curves and the lines $x = 0$ and $y = 0$
- the y-axis, where R is bounded by these curves and the line $x = 0$

The bounds of integration can be given to you in different ways. Make sure you're comfortable interpreting the information the question is giving you, no matter the form. Remember that the same pieces of information are given in most – if not all – questions (i.e., the equation of the curve, the rotation axis, and the bounds), so it's just a matter of knowing what to look out for.

We start by sketching the curves and plugging in (4,32) to f and g to check this is the point of intersection. We can identify the different regions we are asked about on our sketch.	For P we need $f(x) = g(x)$: $x^2 + 16 = 36 - x$ $(4)^2 + 16 = 36 - (4)$ $16 + 16 = 36 - 4$ $32 = 32$ So LHS = RHS as required R(y)  R(x)
We can't use our formulae when our region is bounded by a curve with a changing function – from f to g – so we need to split R into regions where each has a continuous curve. For a rotation about the x-axis, we draw a perpendicular line from P to the x-axis. The two sections have the bounds [0,4] and [4, 36], as g intersects the axis at $x = 36$.	So $V = \pi \int_0^4 (f(x))^2 \ dx + \pi \int_4^{36} (g(x))^2 \ dx$ 
Similarly for our y-axis revolution, we draw a perpendicular line from P to the y-axis. We need to find the points of intersection with the y-axis to know our bounds. However, this time f and g should be rearranged to make x the subject. This would mean $(f^{-1}(y))^2$ and $(g^{-1}(y))^2$ (the inverse functions, $f^{-1}(y) = \sqrt{y-16}$ and $g^{-1}(y) = 36 - y$) would be the integrands.	When $x = 0, f(0) = 16$ and $g(0) = 36$ So, $V = \pi \int_{16}^{32} x^2 \ dy + \pi \int_{32}^{36} x^2 \ dy$ $= \pi \int_{16}^{32} (f^{-1}(y))^2 \ dy + \pi \int_{32}^{36} (g^{-1}(y))^2 \ dy$ $= \pi \int_{16}^{32} y - 16 \ dy + \pi \int_{32}^{36} (36 - x)^2 \ dy$ 

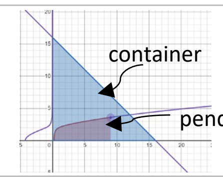
Some revolution problems can be solved more quickly by using the volumes of cylinders and cones:

- For a cylinder of height h and radius r, the volume $V = \pi r^2 h$
- For a cone of height h and radius r, the volume $V = \frac{\pi r^2 h}{3}$

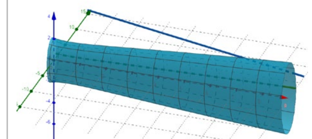
Modelling with volumes of revolution

You may be asked to solve volumes of revolution problems in the context of real life. We can use equations of curves to model what an ideal shape or product would look like in reality. As we are rotating a curve, the objects we will model must be symmetric about an axis. You will be asked to find the volume of this object by rotating it about the x or y axes – which axis you need may be something you have to decide for yourself. The same goes for the bounds of integration. As always, visualising the shapes formed when curves are rotated can help you spot shortcuts – such as identifying cones or cylinders, or where to divide up the region.

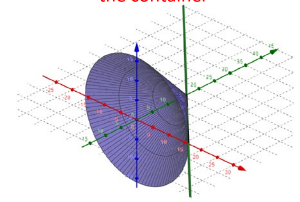
Example 4: A jeweller wants to create a cast of a pendant. They pour the plaster into a container that can be modelled by the region bounded by the lines $y = 16 - x$, $y = 0$, and $x = 0$ that is rotated about the y-axis. The plaster fills in the container around the form of the pendant, which is modelled by revolving the region bounded by the curve $y = \sqrt{x + 4 - \frac{1}{x}}$, the x-axis and the line $x = 9$ about the x-axis. Calculate the amount of plaster needed (assume the container is filled to the brim, and that plaster isn't needed for the volume of the pendant).

As always, we start by sketching out the problem. From our sketch, we can see that to calculate the amount of plaster needed, we need to find the volume of the container and subtract the volume of the pendant.	
When the line $y = 16 - x$ is rotated about the y-axis, we get a cone of height 16, radius 16. In this case, it doesn't make a difference to the volume which axis we rotate about, but this won't always be true, so double check which axis the question asks for. We calculate the volume using the formula for a cone, to save time.	$V_c = \frac{\pi r^2 h}{3} = \pi \times 16^2 \times \frac{16}{3} = \frac{4096}{3}\pi$
We use the formula for a rotation about the x-axis to find the volume of the pendant. This means we must make sure we have y^2, not y. Also, note that the region for the pendant is bounded by the curve, which intersects the x-axis at some value $x > 0$, meaning our lower bound for integration isn't 0 – we need to find the curve's intercept with the axis.	$\sqrt{x + 4 - \frac{1}{x}} = 0 \ \rightarrow \ x^2 + 4x - 1 = 0 \rightarrow x = -2 \pm \sqrt{5}.$ We want the positive root. $V_p = \pi \int_a^b y^2 \ dx = \pi \int_{-2+\sqrt{5}}^{9} x + 4 - \frac{1}{x} \ dx$ $V_p = \pi[\frac{1}{2}x^2 + 4x - \ln x]_{-2+\sqrt{5}}^{9} = \pi(80 - 2\sqrt{5} + \ln(\sqrt{5} - 2) - 2\ln 3)$
Finally, we subtract the volume of the pendant from that of the container to find the amount of plaster needed.	$V_{pl} = V_c - V_p = \pi\left(\frac{3856}{3} + 2\sqrt{5} - \ln(\sqrt{5} - 2) + 2\ln 3\right)$

Resultant volume of revolution for the pendant



Resultant volume of revolution for the container



Sketch problem → Identify axis & recall formula(e) → If needed, divide into regions & identify any cylinders/cones/shortcuts → Get curve equation(s) into correct form → Calculate volumes & find final answer