Roots of polynomials Cheat Sheet

This chapter is concerned with identifying the relationship between the roots of quadratic, cubic and quartic polynomials.

Roots of a quadratic equation

A quadratic equation could have 2 real roots, or 2 complex roots.

- If α and β are roots of the equation $ax^2 + bx + c = 0$, then:
  - $\Rightarrow \alpha + \beta = -\frac{b}{a}$
  - $\Rightarrow \alpha\beta = \frac{c}{a}$

Roots of a cubic equation

A cubic equation could have 3 real roots or 1 real root and 2 complex roots.

- If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:
  - $\Rightarrow \alpha + \beta + \gamma = -\frac{b}{a}$
  - $\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
  - $\Rightarrow \alpha\beta\gamma = -\frac{d}{a}$

Roots of a quartic equation

A quartic equation could have 4 real roots, 4 complex roots or 2 real and 2 complex roots.

- If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:
  - $\Rightarrow \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
  - $\Rightarrow \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
  - $\Rightarrow \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
  - $\Rightarrow \alpha\beta\gamma\delta = \frac{e}{a}$

You can use the following abbreviations to simplify things:

$\sum \alpha = -\frac{b}{a}, \sum \alpha\beta = \frac{c}{a}, \sum \alpha\beta\gamma = -\frac{d}{a}$

Expressions relating to the roots of a polynomial

You can use the following rules to quickly find the values of some expressions concerning the roots of a polynomial:

- Reciprocals:
  - Quadratic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$
  - Cubic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
  - Quartic: $\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$

You may decide not to remember these results since they are quite easy to prove; in each case you simply need to combine the fractions to achieve the *RHS*.

- Products of powers:
  - Quadratic: $\Rightarrow \alpha^n \times \beta^n = (\alpha\beta)^n$
  - Cubic: $\Rightarrow \alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$
  - Quartic: $\Rightarrow \alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

- Rules for sums of squares:
  - Quadratic: $\Rightarrow \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$
  - Cubic: $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$
  - Quartic: $\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

In general, you can remember that $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\left(\sum \alpha\beta\right)$

- Rules for sums of cubes:
  - Quadratic: $\Rightarrow \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$
  - Cubic: $\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = (\alpha+\beta+\gamma)^3 - 3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha) + 3\alpha\beta\gamma$

You won't need to know the result for a quartic polynomial.

Example 1: The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{-1+i}{2}$ and $\beta = \frac{-1-i}{2}$. Find integer values for a, b and c.

Using $\alpha + \beta = -\frac{b}{a}$	$\alpha + \beta = \frac{-1+i}{2} + \frac{-1-i}{2} = -1 = -\frac{b}{a}$
Simplifying:	$\therefore a = b$
Using $\alpha\beta = \frac{c}{a}$	$\alpha\beta = \left(\frac{-1+i}{2}\right)\left(\frac{-1-i}{2}\right) = \frac{1}{2} = \frac{c}{a}$
Simplifying:	$\therefore \frac{1}{2}a = c$
Rewriting the quadratic with $b = a$ and $c = \frac{1}{2}a$: Dividing through by a:	$ax^2 + ax + \frac{1}{2}a = 0$ $x^2 + x + \frac{1}{2} = 0$
Multiplying by 2 to ensure that the constant is an integer:	$2x^2 + 2x + 1 = 0$ $a = 2, b = 2, c = 1$

Example 2: The equation $mx^2 + 4x + 4m = 0$ has roots of the form k and $2k$. Find the values of m and k.

Using $\alpha + \beta = -\frac{b}{a}$:	$\alpha + \beta = k + 2k = 3k = -\frac{4}{m}$
Simplifying:	$\therefore m = -\frac{4}{3k}$
Using $\alpha\beta = \frac{c}{a}$:	$\alpha\beta = (k)(2k) = 2k^2 = \frac{4m}{m} = 4$
Simplifying and solving for k:	$\therefore k^2 = 2$ So $k = \pm\sqrt{2}$
We have two sets of solutions because k has two possible values. Use $m = -\frac{4}{3k}$ to find the corresponding value of m in each case.	$If\ k = \sqrt{2}, m = -\frac{4}{3\sqrt{2}} = -\frac{2\sqrt{2}}{3}$ $If\ k = -\sqrt{2}, m = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$

Example 3: α, β and γ are roots of the cubic equation $7x^3 - 4x^2 - x + 6 = 0$. Find the values of:
(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta\gamma$ (c) $\alpha^3\beta^3\gamma^3$ (d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Using $\alpha + \beta + \gamma = -\frac{b}{a}$:	$\alpha + \beta + \gamma = -\frac{-4}{7} = \frac{4}{7}$
Using $\alpha\beta\gamma = -\frac{d}{a}$:	$\alpha\beta\gamma = -\frac{6}{7}$
Using $\alpha\beta = \frac{c}{a}$:	$\alpha^3\beta^3\gamma^3 = (\alpha\beta\gamma)^3 = \left(-\frac{6}{7}\right)^3 = -\frac{216}{343}$
Using the reciprocal result for cubics:	$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
Using $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$:	$\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{1}{7}$
Substituting back into the reciprocal result:	$\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-\frac{1}{7}}{-\frac{6}{7}} = \frac{1}{6}$

Example 4: The roots of the equation $x^4 + 2x^2 - x + 3 = 0$ are α, β, γ and δ.
(a) Write down the values of $\sum \alpha, \sum \alpha\beta, \sum \alpha\beta\gamma$ and $\sum \alpha\beta\gamma\delta$.
(b) Hence find the values of:
(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ (ii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ (iii) $(\alpha+1)(\beta+1)(\gamma+1)(\delta+1)$

Using $\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$:	$\alpha + \beta + \gamma + \delta = 0$
Using $\sum \alpha\beta = \frac{c}{a}$	$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{2}{1} = 2$
Using $\sum \alpha\beta\gamma = -\frac{d}{a}$	$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{-1}{1} = 1$
Using $\alpha\beta\gamma\delta = \frac{e}{a}$	$\sum \alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{3}{1} = 3$
$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$	$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\sum \alpha\beta\gamma}{\alpha\beta\gamma\delta} = \frac{1}{3}$
Using $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \left(\sum \alpha\right)^2 - 2\left(\sum \alpha\beta\right) = (0)^2 - 2(2) = -4$

Linear transformations of roots

Given a polynomial (up to the fourth degree), you need to be able to find the equation of a second polynomial whose roots are a linear transformation of the roots of the first.

- If a polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ has roots α, β, γ and δ then the polynomial with roots $g\alpha + h, g\beta + h, g\gamma + h$ and $g\delta + h$, where g and h are real constants, is given by $f\left(\frac{w-h}{g}\right)$.

The same logic follows if the polynomial is cubic or quadratic.

Example 5: The quartic equation $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ has roots α, β, γ and δ. Find equations with integer coefficients that have roots:
(i) $(2\alpha), (2\beta), (2\gamma)$ and (2δ). (ii) $(\alpha - 1), (\beta - 1), (\gamma - 1)$ and $(\delta - 1)$.

(i) If $f(x) = x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ then the new equation will be given by $f\left(\frac{w}{2}\right)$.	$f\left(\frac{w}{2}\right) = 2\left(\frac{w}{2}\right)^4 + 4\left(\frac{w}{2}\right)^3 - 5\left(\frac{w}{2}\right)^2 + 2\left(\frac{w}{2}\right) - 1 = 0$
Simplifying:	$\Rightarrow \frac{1}{8}w^4 + \frac{1}{2}w^3 - \frac{5}{4}w^2 + w - 1 = 0$
Multiplying by 8 to ensure all coefficients are integers:	$\Rightarrow w^4 + 4w^3 - 10w^2 + 8w - 8 = 0$
(ii) If $f(x) = x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ then the new equation will be given by $f(w + 1)$.	$f(w+1) = 2(w+1)^4 + 4(w+1)^3 - 5(w+1)^2 + 2(w+1) - 1 = 0$
Expanding brackets then simplifying:	$\Rightarrow 2(w^4 + 4w^3 + 6w^2 + 4w + 1) + 4(w^3 + 3w^2 + 3w + 1) - 5(w^2 + 2w + 1) + 2w + 2 - 1 = 0$ $\Rightarrow 2w^4 + 12w^3 + 19w^2 + 12w + 2 = 0$