Series Cheat Sheet

You need to be comfortable in expressing a series using sigma notation.

$$\sum_{r=1}^{4}(7-2r) = 5+3+1-1 = 8$$

This tells us the last value of r for our sequence, i.e. our last term will be $7-2(4)=-1$

Inputting $r=1$ into this expression gives us the first term, $r=2$ gives us the second term, and so forth.

This is the value of r where our series starts, i.e. our first term is $7-2(1)=5$.

This series in particular is an arithmetic series which is introduced in Pure Year 2.

You can use the following rules to manipulate expressions involving sigma notation:

- $\sum_{r=1}^{n} kf(r) = k\sum_{r=1}^{n} f(r)$
- $\sum_{r=1}^{n} f(r)+g(r) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$

You can use the following results to evaluate some complicated series:

- $\sum_{r=1}^{n} 1 = n$
- $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$
- $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$
- $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

You will not be given these. You will only be given results for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$.

You can prove these results using the proof by induction method covered in Chapter 8 of Core Pure 1.

To find the sum of a series that does not start at $r=1$, you can instead use the following result:

- $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$

Example 1: (a) Show that $\sum_{r=1}^{n}(7r-4) = \frac{1}{2}n(7n-1)$.
(b) Hence evaluate $\sum_{r=20}^{50}(7r-4)$.

(a) Manipulating the sum:	$\sum_{r=1}^{n}(7r-4) = 7\sum_{r=1}^{n} r - 4\sum_{r=1}^{n} 1$
Using the result $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.	$= 7\left[\frac{n}{2}(n+1)\right] - 4n = \frac{1}{2}n[7(n+1)-8)]$
Simplifying:	$= \frac{1}{2}n[7n-1]$
(b) Using the above result to find the sum of a series that does not start at $r=1$:	$\sum_{r=20}^{50}(7r-4) = \sum_{r=1}^{50}(7r-4) - \sum_{r=1}^{19}(7r-4)$
Using our part (a) result with $n=50$ for the first sum and $n=19$ for the second sum:	$= \frac{1}{2}(50)[7(50)-1] - \frac{1}{2}(19)[7(19)-1]$ $= 7471$

Example 2: Show that $\sum_{r=1}^{n} r(r+3)(2r-1) = \frac{1}{6}n(n+1)(3n^2+an+b)$, where a and b are integers to be found.

Expanding the brackets:	$\sum_{r=1}^{n} r(r+3)(2r-1) = \sum_{r=1}^{n} 2r^3+5r^2-3r$
Manipulating the sum:	$= 2\sum_{r=1}^{n} r^3 + 5\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r$
Using the results for $\sum r, \sum r^2, \sum r^3$:	$= 2\left[\frac{1}{4}n^2(n+1)^2\right] + 5\left[\frac{1}{6}n(n+1)(2n+1)\right] - 3\left[\frac{n}{2}(n+1)\right]$
Simplifying before factoring out $\frac{1}{6}n(n+1)$:	$= \frac{1}{2}n^2(n+1)^2 + \frac{5}{6}n(n+1)(2n+1) - \frac{3n}{2}(n+1)$ $= \frac{1}{6}n(n+1)[3n(n+1)+5(2n+1)-9]$
Simplifying:	$= \frac{1}{6}n(n+1)[3n^2+13n-4]$

Example 3: (a) Show that $\sum_{r=1}^{n}(3r-2)^2 = \frac{1}{2}n(6n^2-3n-1)$.
(b) Hence find any values of n for which $\sum_{r=5}^{n}(3r-2)^2 + 103\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 3n^3$.

(a) Expanding the brackets:	$\sum_{r=1}^{n}(3r-2)^2 = \sum_{r=1}^{n} 9r^2-12r+4$
Manipulating the sum:	$= 9\sum_{r=1}^{n} r^2 - 12\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$
Using the results for $\sum 1, \sum r, \sum r^2$:	$= \frac{9}{6}n(n+1)(2n+1) - 12\left[\frac{n}{2}(n+1)\right] + 4(n)$
Simplifying:	$= \frac{3n}{2}(2n^2+3n+1) - 6n(n+1) + 4n$
Factoring out $\frac{n}{2}$:	$= \frac{n}{2}[6n^2+9n+3-12n-12+8) = \frac{n}{2}[6n^2-3n-1]$
Dealing with the first term to begin with; the sum starts at $r=5$ so we need to use the result $\sum_{r=k}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{k-1} f(r)$	$\sum_{r=5}^{n}(3r-2)^2 = \sum_{r=1}^{n}(3r-2)^2 - \sum_{r=1}^{4}(3r-2)^2$
Using the result from part a:	$\sum_{r=5}^{n}(3r-2)^2 = \frac{n}{2}[6n^2-3n-1] - \frac{4}{2}[6(4)^2-3(4)-1]$ $= 3n^3 - \frac{3n^2}{2} - \frac{n}{2} - 166$
Dealing with the second term: You can manually calculate the sum once you identify the periodic nature of the sum.	$\sum_{r=1}^{20} r\cos\left(\frac{r\pi}{2}\right)$ has a periodic nature since $\cos\left(\frac{r\pi}{2}\right)$ will be zero for odd r and for even r it will either be 1 or -1. Simply by writing out the first few terms, we can see what this sum will be: $\sum_{r=1}^{20} r\cos\left(\frac{r\pi}{2}\right) = 0-1+0+3+0-5+\cdots-18+0+20$
Adding up the terms:	Adding up the terms manually gives $\sum_{r=1}^{28} r\cos\left(\frac{r\pi}{2}\right) = 14$.
Substituting this result back into the given equation:	$3n^3 - \frac{3n^2}{2} - \frac{n}{2} - 166 + 103(14) = 3n^3$
Simplifying gives us a quadratic:	$\frac{3n^2}{2} + \frac{n}{2} - 1276 = 0$
Solving the quadratic using the quadratic formula: You could also factorise this quadratic.	Quadratic formula: $n=29$ or $n=-\frac{88}{3}$. Term number must be a positive integer so $n=29$.