Argand diagrams

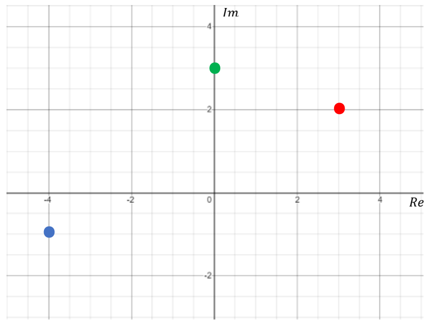
Argand diagrams are used to represent complex numbers. This representation allows us to see the effects of different moduli and arguments, therefore giving us a new way of denoting complex numbers and allows us to solve equations and inequalities graphically using loci.

Argand diagrams

Argand diagrams look similar to Cartesian diagrams – these are the graphs that you are used to seeing, with an x and y axis. As you have seen in the previous chapter, complex numbers have real and imaginary parts. Argand diagrams have the real part of the complex number, denoted Re on what would be the x axis in a Cartesian diagram, and the imaginary part, Im, on the y axis.

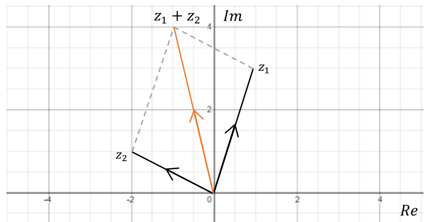
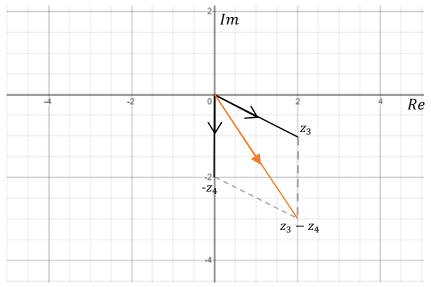
- The complex number $z = x + iy$ can be represented by the point $P(x, y)$ or the vector $\begin{pmatrix} x \\ y \end{pmatrix}$

Example 1: Represent the complex numbers $z_1 = 3 + 2i, z_2 = -4 - i$ and $z_3 = 3i$ on an Argand diagram

The real part of each number tells us the horizontal position and the imaginary part tells us the vertical position, so z_1 is represented by the point that is 3 across and 2 up, shown by the red dot. z_2 and z_3 are denoted by blue and green dots respectively.	

- By using the vector of a complex number, the addition or subtraction of complex numbers can be shown on the argand diagram

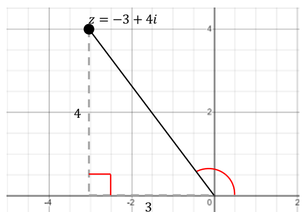
Example 2: For $z_1 = 1 + 3i, z_2 = -2 + i, z_3 = 2 - i$ and $z_4 = 2i$, show $z_1 + z_2$ and $z_3 - z_4$ on an Argand diagram.

For $z_1 + z_2$, plot the vectors of z_1 and z_2 on the Argand diagram. As with adding vectors, place them 'nose-to-tail' and notice that the resultant vector is the diagonal of the parallelogram formed.	
Follow the same process for $z_3 - z_4$, but plot $-z_4$ instead of z_4.	

Modulus and Argument

- The modulus of the complex number $z = x + iy$, denoted $|z|$, is the distance from the origin to the point represented by that number on an Argand diagram, and is given by $|z| = \sqrt{x^2 + y^2}$.
- The argument of a complex number $z = x + iy$, denoted $\arg z$, is the angle $-\pi \le \theta \le \pi$ between the positive real axis and the line joining the point represented by z to the origin. The argument satisfies $\tan\theta = \frac{y}{x}$.

Example 3: Find the modulus and argument of the complex number $z = -3 + 4i$

Mark the point represented by z and draw the line connecting z to the origin.	

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To find $\|z\|$ we need to find the length of the line connecting the origin and z, which can clearly be done by Pythagoras' theorem.	$\|z\| = \sqrt{3^2 + 4^2} = 5$
To find $\arg z$, we need to find the angle from the positive real axis to the line connecting the origin and z. Clearly the angle from the positive to the negative real axis is π radians, so the angle we are looking for is $\pi -$ (the angle between the line and the negative real axis).	$\arg z = \pi - (\tan^{-1}\frac{4}{3})$ $\arg z = 2.214$ rad

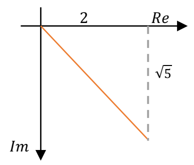
It is important to make sure that you find the right angle for the argument- it is not always $\tan^{-1}\frac{y}{x}$. To ensure that the correct angle is calculated, it is recommended to draw a quick sketch.

Modulus-argument form of complex numbers

- For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus argument form of z is

$$z = r(\cos\theta + i\sin\theta)$$

Example 4: Express the complex number $w = 2 - \sqrt{5}i$ in modulus-argument form

Draw a sketch of the complex number on the diagram.	
Find the modulus, r, and the argument θ.	$r = \sqrt{2^2 + (-\sqrt{5})^2} = 3$ $\theta = \tan^{-1}(\frac{-\sqrt{5}}{2}) = -0.841$
Put into modulus-argument form.	$w = 3(\cos(-0.841) + i\sin(-0.841))$

Multiplying and dividing complex numbers in modulus argument form is simple using certain results:

- For any two complex numbers z_1 and z_2:

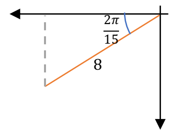
$$|z_1 z_2| = |z_1||z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

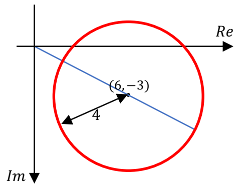
Example 5: For $z_1 = 2(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})$ and $z_2 = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, find $w = z_1 z_2$ in the forms $r(\cos\theta + i\sin\theta)$ and $x + iy$.

Use the rules for moduli and arguments to find the modulus and argument of $w = z_1 z_2$.	$\|w\| = 2 \times 4 = 8$ $\arg(w) = \frac{4\pi}{5} + \frac{\pi}{3} = \frac{17\pi}{15}$ As $-\pi \le \theta \le \pi$, $\arg(w) = -\frac{13\pi}{15}$.
As we have found the modulus and argument of w, it can be put into the form $r(\cos\theta + i\sin\theta)$.	$w = 8(\cos -\frac{13\pi}{15} + i\sin -\frac{13\pi}{15})$
To find w in the form $x + iy$ without a graphical calculator, it can be helpful to draw a quick sketch and use trigonometry. As the angle between the positive real axis and the line from the origin to the complex number is $\frac{-13\pi}{15}$, then the angle from the negative real axis to the line is $\frac{2\pi}{15}$. Otherwise, simply put the expression in a graphical calculator or use known trig expressions to simplify.	 $y = -8\sin\left(\frac{2\pi}{15}\right) = -3.254$ $x = -8\cos\left(\frac{2\pi}{15}\right) = -7.308$ So $w = -7.308 - 3.254i$

Loci and regions in the Argand diagram

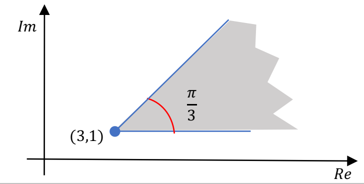
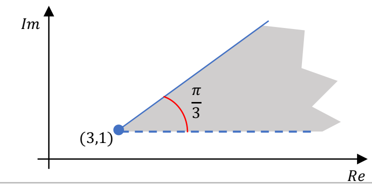
- Complex numbers can be used to represent a locus of points on an Argand diagrams
- Given a complex number $z_1 = x + iy$, the locus of a points z on an Argand diagram such that $|z - z_1| = r$, is a circle with centre (x, y) and radius r.
- Given two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the locus of points z on an Argand diagram such that $z - z_1| = |z - z_2|$ is the perpendicular bisector of the segment of line joining z_1 and z_2
- Given $z_1 = x + iy$, the locus of points z such that $\arg(z - z_1) = \theta$ is a half-line (a line from but not including z_1 that extends infinitely) that makes an angle θ with a line from z_1 that is parallel to the real axis.

Example 6: Given that $|z - 6 + 3i| = 4$, sketch the modulus of z and find the maximum value of $|z|$ in the interval $(-\pi, \pi)$.

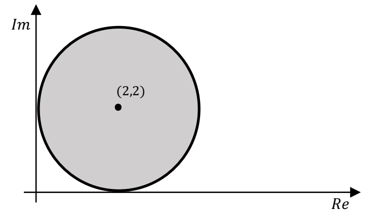
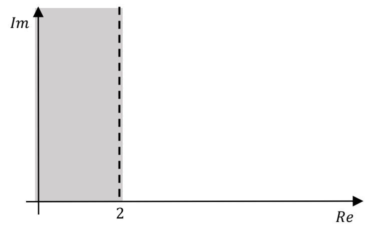
Put the loci equation into the form $\|z - z_1\| = r$.	$\|z - (6 - 3i)\| = 4$ Thus, the loci of z is a circle of radius 4, centred at $(6, -3)$
Sketch the loci.	
As shown by the diagram above, the maximum value of $\|z\|$ is the second intersection of the line from the origin through the centre of the circle, C.	The maximum value of $\|z\|$ is given by $\|OC\| + r$ $\|OC\| = \sqrt{6^2 + (-3)^2} = 3\sqrt{5}$ $\|z\|_{max} = 4 + 3\sqrt{5}$

Complex numbers can also be used to represent regions in the Argand diagram- make sure to pay attention to the inequality signs and any set notation used

Example 7: Sketch the region represented by $0 < \arg(z - 3 - i) \le \frac{\pi}{3}$

Sketch the loci represented by $\arg(z - 3 - i) = \frac{\pi}{3}$ and $\arg(z - 3 - i) = 0$. Therefore, the region represented by $0 \le \arg(z - 3 - i) \le \frac{\pi}{3}$ is the shaded region between the two lines.	
However, the question asks for $0 < \arg(z - 3 - i) \le \frac{\pi}{3}$, so the half line representing $\arg(z - 3 - i) = 0$ should be dashed as the points where the argument is 0 are not included.	

Example 8: Sketch the region represented by $\{|z - 2 - 2i| \le 2 \cap |z - 0.5| < |z - 1.5|\}$

This question involves the regions $\|z - 2 - 2i\| \le 2$ and $\|z - 0.5\| < \|z - 1.5\|$. Sketch the regions on separate graphs. $\|z - 2 - 2i\| \le 2$ represents the interior and boundary line of the circle centred at $2 + 2i$ with radius 2. $\|z - 1.5\| < \|z - 2.5\|$ represents the region where the real value is less than 2. All the points in this region are closer to $(1.5, 0)$ than $(2.5, 0)$.	 
The '$\cap$' symbol is an intersection symbol- we are looking for the set of points that satisfy $\|z - 2 - 2i\| \le 2$ **AND** $\|z - 1.5\| < \|z - 2.5\|$.