Complex Numbers Cheat Sheet

Complex numbers are used by physicists and engineers, specifically in electronics. Complex numbers allow two real quantities to be put together, making the numbers easier to work with, and allowing more complex situations to be modelled.

Complex numbers and imaginary numbers

When solving quadratic equations with the quadratic formula, some equations can't be solved and do not give real solutions. This occurs specifically when the discriminant $\boldsymbol{b^2 - 4ac}$ is less than 0, as the expression under the square root in the quadratic formula is negative, so there are no real solutions. If we extend the number system to include the concept of $\sqrt{-1}$, denoted i, we can represent any solution.

- $i = \sqrt{-1}$
- An **imaginary number** is a number of the form bi, where $b \in \mathbb{R}$.
- A **complex number** can have both real and imaginary parts and is written in the form $z = a + bi$, $a, b \in \mathbb{R}$. $Re(z) = a$ is the real part and $Im(z) = b$ is the imaginary part.

Example 1: Write $\sqrt{-44}$ in terms of i.

Factor out the negative, using rules of surds that you already know.	$\sqrt{-44} = \sqrt{44} \times \sqrt{-1}$
Use the fact that $i = \sqrt{-1}$ to rewrite in terms of i.	$\sqrt{-44} = \sqrt{44}\,i$
Simplify the real surd.	$\sqrt{-44} = \sqrt{4}\sqrt{11}\,i$ $\sqrt{-44} = (2\sqrt{11})i$

When **adding** and **subtracting** complex numbers you must collect 'like' terms. This means real and imaginary terms must be added separately.

Example 2: Simplify the sum $(3 + 4i) + (5 - 6i)$ into the form $a + bi$, where $a, b \in \mathbb{R}$.

Add the real terms.	$(3 + 4i) + (5 - 6i) = 8 + 4i - 6i$
Add the imaginary terms.	$(3 + 4i) + (5 - 6i) = 8 - 2i$

Complex numbers can be **multiplied** by a real number by expanding brackets in the usual way.

Example 3: Evaluate $2(6 + 5i)$

Multiply out the bracket in the usual way, keeping the real and imaginary terms separate.	$2(6 + 5i) = (2 \times 6) + (2 \times 5i)$ $= 12 + 10i$

When multiplying complex numbers, you can multiply out the brackets in the usual way but take into consideration that $i^2 = -1$ and is therefore real.

Example 4: Simplify the product $(3 + 4i)(-5 + 6i)$ into the form $a + bi$, where $a, b \in \mathbb{R}$.

Expand the brackets in the normal way.	$(3 + 4i)(-5 + 6i) = -15 + 18i - 20i + 24i^2$
Simplify the i^2 term and collect like terms.	$(3 + 4i)(-5 + 6i) = -15 + 18i - 20i - 24$ $= -39 - 2i$

Example: Evaluate i^4 and i^5.

Use the fact that $i^2 = -1$ to find i^4.	$i^4 = i \times i \times i \times i$ $= -1 \times -1$ $= 1$
Use the result of i^4 to calculate i^5.	$i^5 = i \times i^4$ $= i \times 1$ $= i$

If you have a graphical calculator, it can compute these expressions for you, but it is very important that you can do it yourself as some questions may include a parameter that you can't put into your calculator.
You can use complex numbers to find the solution to any quadratic equation with real coefficients.

Example 5: Solve the equation $x^2 + 4x + 5 = 0$ using the quadratic formula.

Substitute the values $a = 1, b = 4$ and $c = 5$ into the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2(1)}$
Simplify the expression.	$x = \frac{-4 \pm \sqrt{-4}}{2}$
Put the expression into the form $a + bi$.	$x = -2 \pm \frac{2i}{2} = -2 \pm i$

Example 6: Solve the equation $z^2 - 2z + 17 = 0$ by completing the square

Find an expression for $z^2 - 2z$ by halving the coefficient in front of the z term to find what should go inside the brackets that you are squaring. Expanded, this expression is $z^2 - 2z + 1$, so we must subtract 1 to make the expression equivalent to $z^2 - 2z$.	$z^2 - 2z = (z - 1)^2 - 1$
Adjust the expression to include the constant term.	$z^2 - 2z + 17 = (z - 1)^2 - 1 + 17$ $= (z - 1)^2 + 16$
Set the completed square expression to zero and solve the equation.	$(z - 1)^2 + 16 = 0$ $(z - 1)^2 = -16$ $z - 1 = \pm\sqrt{-16}$ $z - 1 = \pm 4i$ $z = 1 \pm 4i$

Complex conjugation

For a complex number $z = a + bi$, the **complex conjugate** is defined as $\boldsymbol{z^* = a - bi}$.

Example 7: For $z = 3 - 8i$, find z^*, $z + z^*$ and zz^*.

Find z^* by changing the sign of the imaginary part.	$z^* = 3 + 8i$
Evaluate $z + z^*$.	$z + z^* = 3 - 8i + 3 + 8i$ $= 6$
Evaluate zz^*.	$zz^* = (3 - 8i)(3 + 8i)$ $= 9 + 24i - 24i - 64i^2$ $= 73$

Dividing complex numbers is similar to simplifying fractions involving surds – it is not good practice to have an imaginary number on the denominator of a fraction (just like surds), and thus it can be simplified by realising (or rationalising, equivalently for surds) the denominator of the fraction, which can be achieved by using the complex conjugate.

Example 8: Write $\frac{3+2i}{4+5i}$ in the form $a + bi$.

Find the complex conjugate of the denominator.	$(4 + 5i)^* = 4 - 5i$
Multiply the numerator and denominator of the fraction by the complex conjugate and simplify.	$\frac{3 + 2i}{4 + 5i} = \frac{3 + 2i}{4 + 5i} \times \frac{4 - 5i}{4 - 5i}$ $= \frac{12 - 15i + 8i - 10i^2}{16 - 20i + 20i - 25i^2}$ $= \frac{22 - 7i}{41} = \frac{22}{41} - \frac{7}{41}i$

Roots of quadratic equations

If the roots of the quadratic equation $az^2 + bz + c = 0$, with real coefficients, are complex numbers, then they occur as **conjugate pairs**. This means that if z is a root, then z^* must be too. This is useful in finding all of the roots of an equation or finding the original equation itself.

- If the roots of a quadratic equation are α and β, then the equation can be written as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$

Example 9: Given that $\alpha = 6 + 9i$ is a root of a quadratic equation with real roots, state the value of the other root β and find the quadratic equation.

State β, the complex conjugate of α.	$\beta = 6 - 9i$
Either use the product and sum of the roots, or expand the factorised equation to find the equation.	$\alpha + \beta = 12$ $\alpha\beta = 36 - 54i + 54i - 81i^2 = 117$
State the quadratic equation.	$z^2 - 12z + 11 = 0$

Solving cubic and quartic equations

The statements we have made about complex conjugate roots don't only apply for quadratic equations – they also apply for polynomials of higher degree, such as cubics and quartics.

- If $f(z)$ is a polynomial with real coefficients and z_1 is a root of $f(z) = 0$, then z_1^* is also a root of $f(z) = 0$.

Therefore:

- Any cubic equation with real coefficients either has three real roots (that may be repeated) or a complex conjugate pair and one real root.

Example 10: Given that 1 is a root of the equation $z^3 - 7z^2 + kz - 10 = 0$, find the value of k and the other two roots of the equation.

Substitute the value $z = 1$ and evaluate to find the value k.	$(1)^3 - 7(1)^2 + k - 10 = 0$ $k = 10 + 7 - 1$ $k = 16$
Find the roots of the equation by either long division or equating the coefficients of a quadratic. The coefficient of z^3 is 1, so the coefficient of z^2 must also be 1.	$(z - 1)(z^2 + az + b)$ $z^3 + az^2 + bz - z^2 - az - b$ $z^3 + (a - 1)z^2 + (b - a)z - b = 0$ Equating the coefficients: $b = 10$ $a - 1 = -7 \Rightarrow a = -6$
State the quadratic factor of the cubic. The two other roots of the cubic will be the roots of the quadratic.	$z^2 - 6x + 10 = 0$
Use the quadratic formula, or your preferred method, to find the roots of the quadratic.	$z = \frac{6 \pm \sqrt{(-36)^2 - 4(1)(10)}}{2(1)}$ $z = 3 \pm i$
State all the roots of the cubic.	$z = 1, 3 + i, 3 - i$

- An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is a **quartic** with real coefficients. Either all four roots are real (some may be repeated), two roots are real and the other two roots a complex conjugate pair, or there are two sets of complex conjugate pairs as roots – these can be repeated, for example the equation $(z - (2 - 3i))(z - (2 + 3i))(z - (2 - 3i))(z - (2 + 3i)) = 0$

Example 11: Given $1 + 3i$ is a root of the equation $z^4 + 2z^3 - 3z^2 + 50z - 50 = 0$, find the other three roots.

Use the concept of complex conjugate pairs to find another root.	If $1 + 3i$ is a root, then $(1 + 3i)^* = 1 - 3i$ is also a root.
Find a quadratic factor of the quartic using the roots already found.	$(z - (1 + 3i))(z - (1 - 3i)) = z^2 - 2z + 10$
Find the second quadratic factor of the equation, either by long division or equating the coefficients.	$(z^2 - 2z + 10)(z^2 + az + b) = z^4 + 2z^3 - 3z^2 + 50z - 50$ $= z^4 + az^3 + bz^2 - 2z^3 - 2az^2 - 2bz + 10z^2 + 10az + 10b$ $= z^4 + (a - 2)z^3 + (b - 2a + 10)z^2 + (-2b + 10a) + 10b$ $a - 2 = 2 \Rightarrow a = 4$ $10b = -50 \Rightarrow b = -5$ Checking: $b - 2a + 10 = -3$ $-2b + 10a = 50$ So, the second quadratic factor is $z^2 + 4z - 5$.
Find the roots of the second quadratic factor using your preferred method.	$z^2 + 4z - 5 = (z + 5)(z - 1)$ So, $z = -5$ or $z = 1$
State all the roots of the quartic.	$z = 1 + 3i, 1 - 3i, -5$ or 1