AQA Further Maths AS-level

Core

Formula Sheet

Provided in formula book

Not provided in formula book

Complex Numbers

Complex Algebra

For two complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$
$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$
$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{{x_2}^2 + {y_2}^2}$

Complex Conjugation

For: $z = x + iy$	$z^* = x - iy$

$$zz^* = |z|^2$$

Modulus-Argument Form

Modulus of a complex number $z = x + iy$: $\lvert z\rvert = r$	$r = \sqrt{x^2 + y^2}$
Argument of a complex number $z = x + iy$: $\arg(z)$	$\tan^{-1}\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right), 0 \le \theta \le \frac{\pi}{2}$
	$\pi - \tan^{-1}\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right), \frac{\pi}{2} < \theta \le \pi$
	$-\left(\pi - \tan^{-1}\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right)\right), -\pi \le \theta < -\frac{\pi}{2}$
	$-\tan^{-1}\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right), -\frac{\pi}{2} < \theta \le 0$
Modulus-argument form	$z = r(\cos\theta + i\sin\theta)$

Multiplying and Dividing in Modulus-Argument Form

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$	$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert = r_1 r_2$
	$\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert} = \frac{r_1}{r_2}$
	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$
	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$

Loci

Circle with centre $z = a$ and radius r	Outline: $\lvert z - a\rvert = r$
	Interior: $\lvert z - a\rvert < r$
	Exterior: $\lvert z - a\rvert > r$
Perpendicular bisector of points a and b	$\lvert z - a\rvert = \lvert z - b\rvert$
Half-line starting at $z = a$ at an angle of θ from the positive real axis	$\arg(z - a) = \theta$

Matrices

Matrix Arithmetic

$\boldsymbol{M_1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\boldsymbol{M_1} + \boldsymbol{M_2} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$
$\boldsymbol{M_2} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$	$\boldsymbol{M_1} - \boldsymbol{M_2} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$
	$\boldsymbol{M_1} \times \boldsymbol{M_2} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
Let A be a scalar	$A\boldsymbol{M_1} = \begin{bmatrix} Aa & Ab \\ Ac & Ad \end{bmatrix}$

Zero Matrix/ Null Matrix

$$0_n = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Matrix Transformations

Enlargement of scale factor k about O	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Stretch of scale factor k parallel to the x-axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Stretch of scale factor k parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Shear with x-axis fixed	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear with y-axis fixed	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
Anticlockwise rotation through θ about O	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Reflection in the line $y = (\tan\theta)x$	$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
3D rotation through θ about the x-axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
3D rotation through θ about the y-axis	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
3D rotation through θ about the z-axis	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Invariance

Invariant points	$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$
Invariant lines	$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$ where $\begin{bmatrix} x' \\ y' \end{bmatrix}$ is on the same line as $\begin{bmatrix} x \\ y \end{bmatrix}$

Determinant of a Matrix

$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\det(\boldsymbol{A}) = \Delta = ad - bc$

Singularity

$\boldsymbol{A}$ is singular	$\det(\boldsymbol{A}) = \Delta = 0$
$\boldsymbol{A}$ is non-singular	$\det(\boldsymbol{A}) = \Delta \neq 0$

Properties of the Inverse of a Matrix

$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ where $\boldsymbol{A}$ is a non-singular matrix
$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$ where $\boldsymbol{M}$ and $\boldsymbol{N}$ are square matrices of the same order

Calculating the Inverse of a Matrix

$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\boldsymbol{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Further Algebra and Functions

Roots of Polynomials

Quadratic: $ax^2 + bx + c = 0$ Roots: α, β	$\alpha + \beta = -\frac{b}{a}$
	$\alpha\beta = \frac{c}{a}$
Cubic: $ax^3 + bx^2 + cx + d = 0$ Roots: α, β, γ	$\alpha + \beta + \gamma = -\frac{b}{a}$
	$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
	$\alpha\beta\gamma = -\frac{d}{a}$
Quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$ Roots: $\alpha, \beta, \gamma, \delta$	$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
	$\sum \alpha\beta = \alpha\beta + \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
	$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
	$\sum \alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$

Summations

Sum of Natural Numbers	$\sum_{r=1}^{n} 1 = n$
Sum of Integers	$\sum_{r=1}^{n} n = \frac{1}{2}n(n+1)$
Sum of Squares	$\sum_{r=1}^{n} n^2 = \frac{1}{6}n(n+1)(2n+1)$
Sum of Cubes	$\sum_{r=1}^{n} n^3 = \frac{1}{4}n^2(n+1)^2$

Conics

	Ellipse	Parabola	Hyperbola
Standard form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2 = 4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Parametric form	$x = a\cos\theta$ $y = b\sin\theta$	$x = at^2$ $y = 2at$	$x = a\sec\theta$ $y = b\tan\theta$
Asymptotes	None	None	$\frac{x}{a} = \pm\frac{y}{b}$

Transformations of Curves

Translation by vector $\begin{bmatrix} a \\ b \end{bmatrix}$	$f(x,y)=0 \rightarrow f(x-a,y-b)=0$
Stretch	Parallel to x-axis: $f(x,y)=0 \rightarrow f\left(\frac{x}{a},y\right)=0$
	Parallel to y-axis: $f(x,y)=0 \rightarrow f\left(x,\frac{y}{a}\right)=0$
Reflection	In x-axis: $f(x,y)=0 \rightarrow f(x,-y)=0$
	In y-axis: $f(x,y)=0 \rightarrow f(-x,y)=0$
	In $y=x$: $f(x,y)=0 \rightarrow f(y,x)=0$
	In $y=-x$: $f(x,y)=0 \rightarrow f(-y,-x)=0$

Further Calculus

Volumes of Revolution

Rotation about the x-axis	$V=\pi\int_a^b y^2 dx$
Rotation about the y-axis	$V=\pi\int_a^b x^2 dy$

Mean Value of a Function

$$\bar{f}=\frac{1}{b-a}\int_a^b f(x)\,dx$$

Further Vectors

Equation of a Straight Line in 3D

Vector form of a line through $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ parallel to vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$	$r = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \boldsymbol{\lambda} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Cartesian form	$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \boldsymbol{\lambda}$
Vector product form	$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = 0$

Scalar Product

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + c_x c_y = |\boldsymbol{a}||\boldsymbol{b}| \, cos\, \theta$$

Scalar Projection

Resolved part of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$: $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$

Perpendicular Vectors

Two vectors a and b are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Vector Product

$$\mathbf{a} \times \mathbf{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \, \hat{\boldsymbol{n}} = \begin{bmatrix} \boldsymbol{i} & a_1 & b_1 \\ \boldsymbol{j} & a_2 & b_2 \\ \boldsymbol{k} & a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

Area of a Triangle with Sides $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\mathrm{A} = \frac{1}{2} |\boldsymbol{a} \times \boldsymbol{b}|$$

Polar Coordinates

Converting between Polar, $(\boldsymbol{r}, \boldsymbol{\theta})$, and Cartesian, $(\boldsymbol{x}, \boldsymbol{y})$, Coordinates

$r^2 = x^2 + y^2$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ ($\pm\pi$ if needed)
$x = r\cos\theta$
$y = r\sin\theta$

Polar Curves

Circle	$r = a$
Spiral	$r = a\theta$
Half-line	$r = \theta$
Trefoil	$r = a\cos n\theta$
	$r = a\sin n\theta$
Cardioid	$r = a(b + \cos\theta), \lvert b\rvert = 1$
Egg	$r = a(b + \cos\theta), \lvert b\rvert \geq 2$
Egg with dimple	$r = a(b + \cos\theta), 1 < \lvert b\rvert < 2$

Hyperbolic Functions

Definitions

Function	Exponential Form
$\sinh x$	$\dfrac{e^x - e^{-x}}{2}$
$\cosh x$	$\dfrac{e^x + e^{-x}}{2}$
$\tanh x$	$\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Inverse Hyperbolic Functions

Function	Logarithmic Form
$\sinh^{-1} x$	$\ln\left(x + \sqrt{1 + x^2}\right)$
$\cosh^{-1} x$	$\ln\left(x + \sqrt{x^2 - 1}\right) \quad (x \geq 1)$
$\tanh^{-1} x$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (\lvert x\rvert < 1)$

Hyperbolic Identities

$\tanh x \equiv \dfrac{\sinh x}{\cosh x}$
$\cosh^2 x - \sinh^2 x \equiv 1$
$\sinh 2x = 2 \sinh x \cosh x$
$\cosh 2x = \cosh^2 x + \sinh^2 x$