AQA Further Maths A-level

Core

Formula Sheet

Provided in formula book	Not provided in formula book

Complex Numbers

Complex Algebra

For two complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$
$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$
$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{{x_2}^2 + {y_2}^2}$

Complex Conjugation

For $z = x + iy$	$z^* = x - iy$
For $z = r\,e^{i\theta}$	$z^* = r\,e^{-i\theta}$

$$zz^* = |z|^2$$

Modulus-Argument Form

Modulus of a complex number $z = x + iy$: $\|z\| = r$	$r = \sqrt{x^2 + y^2}$
Argument of a complex number $z = x + iy$: $\arg(z)$	$\tan^{-1}\left(\frac{\|y\|}{\|x\|}\right), 0 \leq \theta \leq \frac{\pi}{2}$
	$\pi - \tan^{-1}\left(\frac{\|y\|}{\|x\|}\right), \frac{\pi}{2} < \theta \leq \pi$
	$-\left(\pi - \tan^{-1}\left(\frac{\|y\|}{\|x\|}\right)\right), -\pi \leq \theta < -\frac{\pi}{2}$
	$-\tan^{-1}\left(\frac{\|y\|}{\|x\|}\right), -\frac{\pi}{2} < \theta \leq 0$
Modulus-argument form	$z = r(\cos\theta + i\sin\theta)$

Multiplying and Dividing in Modulus-Argument Form

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$	$\|z_1 z_2\| = \|z_1\|\|z_2\| = r_1 r_2$
	$\left\|\frac{z_1}{z_2}\right\| = \frac{\|z_1\|}{\|z_2\|} = \frac{r_1}{r_2}$
	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$
	$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$

De Moivre's Theorem

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Euler's Formula

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$

n^{th} Roots of Unity

For $z^n = 1$	$z = e^{\frac{2\pi k i}{n}}$ for $k = 0, 1, 2, \dots, n-1$
	$\omega_k = \left(e^{\frac{2\pi i}{n}}\right)^k$ for $k = 0, 1, 2, \dots, n-1$
	$1 + \omega + \omega_2 + \dots + \omega_{n-1} = \frac{(1-\omega_n)}{1-\omega} = 0$

Factorisation

$$(z - re^{i\theta})(z - re^{-i\theta}) = z^2 - 2r\cos\theta + r^2$$

Loci

Circle with centre $z = a$ and radius r	Outline: $\lvert z - a\rvert = r$
	Interior: $\lvert z - a\rvert < r$
	Exterior: $\lvert z - a\rvert > r$
Perpendicular bisector of points a and b	$\lvert z - a\rvert = \lvert z - b\rvert$
Half-line starting at $z = a$ at an angle of θ from the positive real axis	$\arg(z - a) = \theta$

Powers of Trigonometric Functions

For $e^{i\theta} = \cos\theta + i\sin\theta$	$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - \sin\theta$
	$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$
	$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
For $z = e^{i\theta}$	$z^n + \frac{1}{z^n} = 2\cos n\theta$
	$z^n - \frac{1}{z^n} = 2i\sin n\theta$

Matrices

Matrix Arithmetic

$\boldsymbol{M_1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\boldsymbol{M_2} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ Let A be a scalar	$\boldsymbol{M_1} + \boldsymbol{M_2} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$
	$\boldsymbol{M_1} - \boldsymbol{M_2} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$
	$\boldsymbol{M_1} \times \boldsymbol{M_2} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$
	$A\boldsymbol{M_1} = \begin{bmatrix} Aa & Ab \\ Ac & Ad \end{bmatrix}$

Zero Matrix / Null Matrix

$$0_n = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Identity Matrix

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Matrix Transformations

Enlargement of scale factor k about O	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Stretch of scale factor k parallel to the x-axis	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Stretch of scale factor k parallel to the y-axis	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
Reflection in the x-axis	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection in the y-axis	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection in the line $y = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection in the line $y = -x$	$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
Shear with x-axis fixed	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Shear with y-axis fixed	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
Anticlockwise rotation through θ about O	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Reflection in the line $y = (\tan\theta)x$	$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
3D rotation through θ about the x-axis	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
3D rotation through θ about the y-axis	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
3D rotation through θ about the z-axis	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Invariance

Invariant points	$M\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$
Invariant lines	$M\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x'\\y'\end{bmatrix}$ Where $\begin{bmatrix}x'\\y'\end{bmatrix}$ is on the same line as $\begin{bmatrix}x\\y\end{bmatrix}$

Vector Product

$\boldsymbol{a}\times\boldsymbol{b}=\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta\,\hat{n}$	where θ is the angle between vectors a and b and $\hat{n}$ is a unit vector perpendicular to a and b
$\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$	If $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$

Properties of Vector Products

$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$
$(k\boldsymbol{a})\times\boldsymbol{b}=k(\boldsymbol{a}\times\boldsymbol{b})$
$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$
$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})$
$\boldsymbol{a}\times\boldsymbol{a}=\begin{pmatrix}a_2a_3-a_3a_2\\a_3a_1-a_1a_3\\a_1a_2-a_2a_1\end{pmatrix}=0$
If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a}\times\boldsymbol{b}=0$

Equation of line passing through the point with position vector $\boldsymbol{a}$ and parallel to vector $\boldsymbol{d}$	$(\boldsymbol{r}-\boldsymbol{a})\times\boldsymbol{d}=\boldsymbol{0}$
Area of triangle with sides $\boldsymbol{a}$ and $\boldsymbol{b}$	$\frac{1}{2}\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert$
Vector equation of plane parallel to vectors $\boldsymbol{d_1}$ and $\boldsymbol{d}_2$ containing point $\boldsymbol{a}$	$\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{d}_1+\mu\boldsymbol{d}_2$
Scalar product equation of a plane	$\boldsymbol{r}\cdot\boldsymbol{n}=\boldsymbol{a}\cdot\boldsymbol{n}$ Where $\boldsymbol{n}=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ is normal to the plane

Determinant of a Matrix

$\boldsymbol{A}=\begin{bmatrix}a & b\\c & d\end{bmatrix}$	$\det(\boldsymbol{A})=\Delta=ad-bc$
$\boldsymbol{A}=\begin{bmatrix}a & b & c\\d & e & f\\g & h & i\end{bmatrix}$	$\det(\boldsymbol{A})=\Delta=a\begin{vmatrix}e & f\\h & i\end{vmatrix}-b\begin{vmatrix}d & f\\g & i\end{vmatrix}+c\begin{vmatrix}d & e\\g & h\end{vmatrix}$ $=a(ei-fh)-b(di-fg)+c(dh-eg)$

Singularity

$\boldsymbol{A}$ is singular	$\det(\boldsymbol{A}) = \Delta = 0$
$\boldsymbol{A}$ is non-singular	$\det(\boldsymbol{A}) = \Delta \neq 0$

Transpose of a Matrix

$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
$(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$

Properties of the Inverse of a Matrix

$\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$ where $\boldsymbol{A}$ is a non-singular matrix
$(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$ where $\boldsymbol{M}$ and $\boldsymbol{N}$ are square matrices of the same order

Calculating the Inverse of a Matrix

$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\boldsymbol{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{C}^T$ $= \frac{1}{\det(\boldsymbol{A})}\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$

Triangular Matrix

Upper Triangular Matrix	$\boldsymbol{U}_n = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$
Lower Triangular Matrix	$\boldsymbol{L}_n = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$

Determinant of a Triangular Matrix

For L_n or U_n	$\Delta = a_{11} \times a_{22} \times \dots \times a_{nn}$

Factorisation of Determinants Using Row and Column Operations

Operation to obtain matrix N from matrix M	Change to determinant
Swapping two rows or columns	$\det N = -\det M$
Multiplying one row or column by constant k	$\det N = k \det M$
Adding a multiple of one row (or column) to another row (or column)	$\det N = \det M$

Characteristic Equation

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$$

Diagonalisation of a Square Matrix

$$\boldsymbol{M} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$$

Raising a Matrix to a Power

$$\boldsymbol{M}^n = \boldsymbol{U}\boldsymbol{D}^n\boldsymbol{U}^{-1}$$

Further Algebra and Functions

Roots of Polynomials

Quadratic: $ax^2 + bx + c = 0$ Roots: α, β	$\alpha + \beta = -\frac{b}{a}$
	$\alpha\beta = \frac{c}{a}$
Cubic: $ax^3 + bx^2 + cx + d = 0$ Roots: α, β, γ	$\alpha + \beta + \gamma = -\frac{b}{a}$
	$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
	$\alpha\beta\gamma = -\frac{d}{a}$
Quartic: $ax^4 + bx^3 + cx^2 + dx + e = 0$ Roots: $\alpha, \beta, \gamma, \delta$	$\sum \alpha = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$
	$\sum \alpha\beta = \alpha\beta + \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
	$\sum \alpha\beta\gamma = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
	$\sum \alpha\beta\gamma\delta = \alpha\beta\gamma\delta = \frac{e}{a}$

Summations

Sum of Natural Numbers	$\sum_{r=1}^{n} 1 = n$
Sum of Integers	$\sum_{r=1}^{n} n = \frac{1}{2}n(n+1)$
Sum of Squares	$\sum_{r=1}^{n} n^2 = \frac{1}{6}n(n+1)(2n+1)$
Sum of Cubes	$\sum_{r=1}^{n} n^3 = \frac{1}{4}n^2(n+1)^2$

Maclaurin Series

$e^x = \exp(x)$	$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots$ for all x
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{r+1}\frac{x^r}{r} + \cdots$ $(-1 < x \leq 1)$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$ for all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^r \frac{x^{2r}}{(2r)!} + \cdots$ for all x
$(1+x)^n$	$1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$ $(\lvert x \rvert < 1, n \in \mathbb{Q})$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^r}{r!}f^r(0)$$

L'Hôpital's Rule

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Conics

	Ellipse	Parabola	Hyperbola
Standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$y^2 = 4ax$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Parametric form	$x = a\cos\theta$ $y = b\sin\theta$	$x = at^2$ $y = 2at$	$x = a\sec\theta$ $y = b\tan\theta$
Asymptotes	None	None	$\frac{x}{a} = \pm\frac{y}{b}$

Transformations of Curves

Translation by vector $\begin{bmatrix} a \\ b \end{bmatrix}$	$f(x,y)=0 \to f(x-a,y-b)=0$
Stretch	Parallel to x-axis: $f(x,y)=0 \to f\left(\frac{x}{a},y\right)=0$
	Parallel to y-axis: $f(x,y)=0 \to f\left(x,\frac{y}{a}\right)=0$
Reflection	In x-axis: $f(x,y)=0 \to f(x,-y)=0$
	In y-axis: $f(x,y)=0 \to f(-x,y)=0$
	In $y=x$: $f(x,y)=0 \to f(y,x)=0$
	In $y=-x$: $f(x,y)=0 \to f(-y,-x)=0$
Rotation anti-clockwise About O	$90°$: $f(x,y)=0 \to f(y,-x)=0$
	$180°$: $f(x,y)=0 \to f(-x,-y)=0$
	$270°$: $f(x,y)=0 \to f(-y,x)=0$
Enlargement by scale factor a, centre O	$f(x,y)=0 \to f\left(\frac{x}{a},\frac{y}{a}\right)=0$

Further Calculus

Volumes of Revolution

Rotation about the x-axis	$V = \pi \int_a^b y^2 dx$
Rotation about the y-axis	$V = \pi \int_a^b x^2 dy$

Mean Value of a Function

$$\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Calculus with Inverse Trigonometric and Hyperbolic Functions

$f(x)$	$f'(x)$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$ $(\lvert x \rvert < a)$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Arc Length of a Curve

Cartesian coordinates	$s = \int_a^b \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$
Parametric coordinates	$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$

Area of a Surface of Revolution

Cartesian coordinates	$s_x = 2\pi \int_a^b y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$
Parametric coordinates	$s_x = 2\pi \int_{t_1}^{t_2} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$

Extremal Limits

$\lim_{x \to \infty} x^k e^{-x} = 0$ where $k > 0$
$\lim_{x \to 0} x^k ln x = 0$ where $k > 0$

Further Vectors

Equation of a Straight Line in 3D

Vector form of a line through $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ parallel to vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$	$r = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \boldsymbol{\lambda} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Cartesian form	$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3} = \boldsymbol{\lambda}$
Vector product form	$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = 0$

Equation of a Plane in 3D

Vector form	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}_1 + \mu \boldsymbol{d}_2$ For the plane containing point $\boldsymbol{a}$ and parallel to directions $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$
Cartesian form	$\boldsymbol{n}_1 x + \boldsymbol{n}_2 y + \boldsymbol{n}_3 z = d$
	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$

Scalar Product

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + c_x c_y = |\boldsymbol{a}||\boldsymbol{b}| \cos \theta$$

Scalar Projection

Resolved part of $\boldsymbol{a}$ in direction of $\boldsymbol{b}$: $\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$

Angle Between Two Planes

$$\cos\theta = \left|\frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right|$$

Angle Between Line and Plane

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta = \left|\frac{\boldsymbol{b}\cdot\boldsymbol{n}}{|\boldsymbol{b}||\boldsymbol{n}|}\right|$$

Perpendicular Vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$

Vector Product

$$\mathbf{a}\times\mathbf{b} = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta\,\hat{\boldsymbol{n}} = \begin{bmatrix} \boldsymbol{i} & a_1 & b_1 \\ \boldsymbol{j} & a_2 & b_2 \\ \boldsymbol{k} & a_3 & b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Area of a Triangle

$$\mathrm{A} = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$$

Polar Coordinates

Converting between Polar, (r, θ), and Cartesian, (x, y), Coordinates

$r^2 = x^2 + y^2$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ ($\pm\pi$ if needed)
$x = r\cos\theta$
$y = r\sin\theta$

Polar Curves

Circle	$r = a$
Spiral	$r = a\theta$
Half-line	$r = \theta$
Trefoil	$r = a\cos n\theta$ $r = a\sin n\theta$
Cardioid	$r = a(b + \cos\theta), \lvert b \rvert = 1$
Egg	$r = a(b + \cos\theta), \lvert b \rvert \geq 2$
Egg with dimple	$r = a(b + \cos\theta), 1 < \lvert b \rvert < 2$

Area Enclosed by a Polar Curve

$$A = \frac{1}{2}\int r^2 \, d\theta$$

Hyperbolic Functions

Hyperbolic and Reciprocal Hyperbolic Functions

Function	Exponential Form	Domain	Range
$\sinh x$	$\dfrac{e^x - e^{-x}}{2}$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
$\cosh x$	$\dfrac{e^x + e^{-x}}{2}$	$x \in \mathbb{R}$	$f(x) \geq 1$
$\tanh x$	$\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$x \in \mathbb{R}$	$-1 < f(x) < 1$
$\operatorname{cosech} x = \dfrac{1}{\sinh x}$	$\dfrac{e^x - e^{-x}}{2}$	$x \neq 0$	$f(x) \neq 0$
$\operatorname{sech} x = \dfrac{1}{\cosh x}$	$\dfrac{2}{e^x + e^{-x}}$	$x \in \mathbb{R}$	$f(x) > 0$
$\coth x = \dfrac{1}{\tanh x}$	$\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$	$x \neq 0$	$f(x) < -1$, $f(x) > 1$

Differentiation

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$

Integration

$f(x)$	$\int f(x)\,\mathrm{d}x$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\tanh x$	$ln \cosh x + c$
$\frac{1}{\sqrt{a^2 + x^2}}$	$\sinh^{-1}\left(\frac{x}{a}\right) + c$ or $ln\left(x + \sqrt{x^2 + a^2}\right) + c$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\cosh^{-1}\left(\frac{x}{a}\right) + c$ or $ln\left(x + \sqrt{x^2 - a^2}\right) + c$ $(x > a)$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} ln\left(\frac{a + x}{a - x}\right) = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c$ $(\lvert x \rvert < a)$
$\frac{1}{x^2 - a^2}$	$\frac{1}{2a} ln\left(\frac{x - a}{x + a}\right) + c$

Inverse Hyperbolic Functions

Function	Logarithmic Form	Domain	Range
$\sinh^{-1} x$	$ln\left(x + \sqrt{1 + x^2}\right)$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
$\cosh^{-1} x$	$ln\left(x + \sqrt{1 - x^2}\right)$	$x \geq 1$	$f(x) \geq 0$
$\tanh^{-1} x$	$\frac{1}{2} ln\left(\frac{1 + x}{1 - x}\right)$	$\|x\| < 1$	$f(x) \in \mathbb{R}$

Hyperbolic Identities

$\tanh x \equiv \frac{\sinh x}{\cosh x}$
$\text{sech}^2\, x \equiv 1 - \tanh^2 x$
$\text{cosech}^2 x \equiv \coth^2 x - 1$
$\cosh^2 x - \sinh^2 x \equiv 1$
$\sinh 2x \equiv 2\sinh x \cosh x$
$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

Differential Equations

Integrating Factor

$$I(x) = e^{\int P(x)\,\mathrm{d}x}$$

Auxiliary Equation

$$\lambda^2 + a\lambda + b = 0$$

General Solution

Solution to Auxiliary Equation	General Solution to Differential Equation
Two distinct real roots λ_1 and λ_2	$y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
Repeated roots λ	$y = (A + Bx)e^{\lambda x}$
Complex roots $\alpha + i\beta$	$y = e^{\alpha x}(A\sin(\beta x) + B\cos(\beta x))$

Particular Integrals

$f(x)$	Trial Function
$ax + b$	$px + q$
Polynomial of order n	$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$
$\cos px$ and/or $\sin px$	$a\cos px + b\sin px$
e^{px}	ae^{px}

Discriminant

$\Delta = b^2 - 4ac > 0$	Distinct real roots, λ_1, λ_2
$\Delta = b^2 - 4ac = 0$	Repeated roots, λ
$\Delta = b^2 - 4ac < 0$	Complex roots, $\lambda_1 = x_1 + iy_1, \lambda_2 = x_2 + iy_2$

Simple Harmonic Motion

$\ddot{x} + \omega^2 x = 0$
$v^2 = \omega^2(a^2 - x^2)$
$\omega = 2\pi f$
$x = A\cos\omega t + B\sin\omega t$

Drag Force

$D = -Kv$

Damping

Light	Complex roots, $\alpha + i\beta$
	$x = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
Critical	Repeated roots, λ
	$x = (A + Bt)e^{\lambda t}$
Heavy	Two distinct real roots, λ_1, λ_2
	$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$

Hooke's Law

$T = kx$

Numerical Methods

Mid-Ordinate Rule

$$\int_a^b y\,dx \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \cdots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right)$$

where $h = \frac{b-a}{n}$

Simpson's Rule

$$\int_a^b y\,dx \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\}$$

where $h = \frac{b-a}{n}$ and n is even

Euler's Method

For $\frac{dy}{dx} = f(x)$, h is small	Euler's method: $y_{n+1} = y_n + hf(x_n)$, $x_{n+1} = x_n + h$
For $\frac{dy}{dx} = f(x, y)$	Euler's method: $y_{r+1} = y_r + hf(x_r, y_r)$, $x_{r+1} = x_r + h$
	Improved Euler method: $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$, $x_{r+1} = x_r + h$