Proof Cheat Sheet

AQA A Level Further Maths: Core

Proof by Induction

Proof by induction is a robust and diverse method of mathematical proof used when the result or final expression is already known. In AQA A-Level Further Mathematics, it is involved only in proving sums of series, divisibility, and powers of matrices.

The four-stage process is always as follows:

1. **Base case:** Prove the result is true for $n = 1$ (or 0).
2. **Assumption:** Assume the result is true for $n = k$.
3. **Inductive step:** Prove the result is true for $n = k + 1$.
4. **Conclusion:** Write a strict concluding statement that the result holds for all positive integers.

This method of proof is often likened to a row of toppling dominos. By proving the result is true for $n = 1$ and $n = k + 1$, it can be deduced that the result must hold for $n = 2$. As it holds for $n = 2$, it must also hold for $n = 3, n = 4, \dots$. Hence, the statement holds for all integers $n \geq 1$.

Sums of Series

Proof by induction is often useful in proving results about sums of series, typically with sigma notation. This includes the standard summation results introduced in the *Further Algebra and Functions* section of the course, which are also given in the data booklet. An example of proof by induction for one of the standard results is shown below.

Example 1: Prove $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ for all positive integer values of n.

It is useful to first work out the 'target expression' for $n = k + 1$, which will be derived later in the inductive step.	Target expression: $\sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}((k+1)+1)(2(k+1)+1)$ $\sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}(k+2)(2k+3)$
Calculate the RHS and LHS separately to show that the result is true for $n = 1$.	LHS: $(1)^2 = 1$ RHS: $\frac{1}{6}(1+1)(2\times 1+1) = 1$ RHS = LHS ∴ Result is true for $n = 1$.
State the assumption that the result is true for $n = k$ by substituting k for n in the whole expression.	Assume true for $n = k$: $\therefore \sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$
Using the result for $n = k$, calculate the result for $n = k + 1$. This is done by adding on the $(k+1)^{th}$ term. Note that it is generally easier to factorise out terms to simplify, rather than expanding every bracket. Keep in mind we are trying to reach the target expression calculated in the first line of working.	$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$ $\sum_{r=1}^{k+1} r^2 = \frac{k}{6}(k+1)(2k+1) + (k+1)^2$ $\sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}[k(2k+1) + 6(k+1)]$ $\sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}[2k^2 + 7k + 1]$ $\sum_{r=1}^{k+1} r^2 = \frac{k+1}{6}(k+2)(2k+3)$ ∴ Result is true for $n = k + 1$.
Write the conclusion. The bolded words **must** be included to emphasise the inductive step.	**If** the statement is true for $n = k$, **then** it is true for $n = k + 1$. **Since** the statement is true for $n = 1$, it is true for all $n \in \mathbb{Z}^+$.

This denotes all n belonging to the set of positive integers.

Example 2: Prove by induction that for all $n \in \mathbb{Z}^+$, $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

Calculate the target expression to be derived in the inductive step. This is found by substituting $k + 1$ into the RHS of the result.	Target expression: $\frac{k+1}{2(k+1)+1} = \frac{k+1}{2k+3}$
Compare the RHS and LHS separately to show that the result is true for $n = 1$.	LHS: $\frac{1}{1\times3} = \frac{1}{3}$ RHS: $\frac{1}{(2\times1)+1} = \frac{1}{3}$ RHS = LHS ∴ Result is true for $n = 1$.
State the assumption that the result is true for $n = k$ by substituting k for n in the whole expression. Note S_k denotes the sum of the first k terms.	Assume true for $n = k$: $S_k = \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$
Using the result for S_k, calculate the result for S_{k+1} (the sum of the first $k + 1$ terms). This is done by adding on the $(k+1)^{th}$ term. Keep in mind we are trying to reach the target expression.	$S_{k+1} = S_k + u_{k+1}$ $S_{k+1} = \frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$ $S_{k+1} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$ $= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$ $= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$ $= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$ $= \frac{k+1}{2k+3}$, which is the target expression. ∴ Result is true for $n = k + 1$.
Write the conclusion.	**If** the statement is true for $n = k$, **then** it is true for $n = k + 1$. **Since** the statement is true for $n = 1$, it is true for all $n \in \mathbb{Z}^+$.

Divisibility

Proving that an expression is divisible by an integer is an area of number theory known as divisibility. There are several methods of proving divisibility by induction; two will be explored in this section. Note that there is no target expression in this type of proof by induction, rather, we try to manipulate the expression for u_{k+1} to show the divisibility property.

Example 3: Prove $6^n + 4$ is divisible by 5 for all integers $n \geq 0$.

Show the result is true for the base case by letting $n = 0$. Then factorise 5 out the expression.	$6^0 + 4 = 1 + 4 = 5 = 5 \times 1$ ∴ Result is true for $n = 0$.
State the assumption that the result is true for $n = k$; this means assuming $6^n + 4$ is a multiple of 5 for all $k \geq 0$. Note u_k denotes the k^{th} term.	Assume true for $n = k$: $u_k = 6^k + 4 = 5m$
Write the result for $n = k + 1$ (denoted by u_{k+1}). By rearranging u_k, rewrite u_{k+1} to show that u_{k+1} is also divisible by 5.	$u_{k+1} = 6^{k+1} + 4$ $6(6^k) + 4$ $6^k + 4 = 5m \Rightarrow 6^k = 5m - 4$ $\therefore 6(6^k) + 4 = 6(5m-4) + 4$ $= 30m - 20$ $= 5(6m - 4)$ ∴ Result is true for $n = k + 1$.
Write the conclusion.	**If** the statement is true for $n = k$, **then** it is true for $n = k + 1$. **Since** the statement is true for $n = 0$, it is true for all $n \geq 0 \in \mathbb{Z}$.

Divisibility problems can be extended by introducing multiple indices. In these cases, it may be easier to prove the statement is true for $n = k + 1$ by subtracting (or adding) multiples of terms from each other. Given one term that is divisible by some integer p, if the difference between this term and another is also divisible by p, both terms must be divisible by p. An example of this is shown below.

Example 4: Prove $13^{n+1} - 7^n$ is divisible by 6 for all integers $n \geq 0$.

Show that the result is true for $n = 0$ by letting $n = 0$. Then factor 6 out the expression.	$13^{0+1} - 7^0 = 13 - 1 = 12 = 6 \times 2$ ∴ Result is true for $n = 1$.
State the assumption that the result is true for $n = k$, i.e., assume $13^{n+1} - 7^n$ is a multiple of 6 for all $k \geq 0$.	Assume true for $n = k$: $u_k = 13^{k+1} - 7^k = 6m$
Write the result for u_{k+1}. Subtract a multiple of u_k to eliminate one of the terms. Here, it can be seen that 7^{k+1} can be eliminated by subtracting $7u_k$.	$u_{k+1} = 13^{(k+1)+1} - 7^{k+1} = 13^{k+2} - 7^{k+1}$ $u_{k+1} - 7u_k = 13^{k+2} - 7^{k+1} - 7(13^{k+1} - 7^k)$ $= 13^{k+2} - 7^{k+1} - 7(13^{k+1}) + 7^{k+1}$ $= 13^{k+2} - 7(13^{k+1})$ $= 13(13^{k+1}) - 7(13^{k+1})$ $= 6 \times 13^{k+1}$ ∴ Result is true for $n = k + 1$.
Write the conclusion.	**If** the statement is true for $n = k$, **then** it is true for $n = k + 1$. **Since** the statement is true for $n = 0$, it is true for all $n \geq 0 \in \mathbb{Z}$.

Powers of Matrices

From the *Matrices* section of the course, it will have been shown that $\boldsymbol{M}^k\boldsymbol{M} = \boldsymbol{M}^{k+1}$. This fact means that matrix multiplication can be used to prove results involving powers of matrices by induction.

Example 5: $\boldsymbol{M} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$. Prove by induction that $\boldsymbol{M}^n = \begin{pmatrix} 2^n & 2^{n+1}-2 \\ 0 & 1 \end{pmatrix}$ for $n \geq 1 \in \mathbb{Z}$.

Calculate the target expression for $n = k + 1$ to be derived later in the inductive step.	Target expression: $\boldsymbol{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 2^{(k+1)+1}-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^{k+2}-2 \\ 0 & 1 \end{pmatrix}$
Show that the result is true for $n = 1$ by substituting 1 for n in the general expression.	$\boldsymbol{M}^1 = \begin{pmatrix} 2^1 & 2^{1+1}-2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ∴ Result is true for $n = 1$.
State the assumption that the result is true for $n = k$ by replacing n with k in the general expression.	Assume true for $n = k$: $\boldsymbol{M}^k = \begin{pmatrix} 2^k & 2^{k+1}-2 \\ 0 & 1 \end{pmatrix}$
Prove the result is true for $n = k + 1$ by multiplying the result for $\boldsymbol{M}^k$ with $\boldsymbol{M}$.	$\boldsymbol{M}^k \times \boldsymbol{M} = \begin{pmatrix} 2^k & 2^{k+1}-2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 2^{k+1} & 2^{k+1}+2^{k+1}-2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 2^{k+1} & 2(2^{k+1})-2 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 2^{k+1} & 2^{k+2}-2 \\ 0 & 1 \end{pmatrix}$, which is the target expression. ∴ Result is true for $n = k + 1$.
Write the conclusion.	**If** the statement is true for $n = k$, **then** it is true for $n = k + 1$. **Since** the statement is true for $n = 1$, it is true for all $n \in \mathbb{Z}^+$.