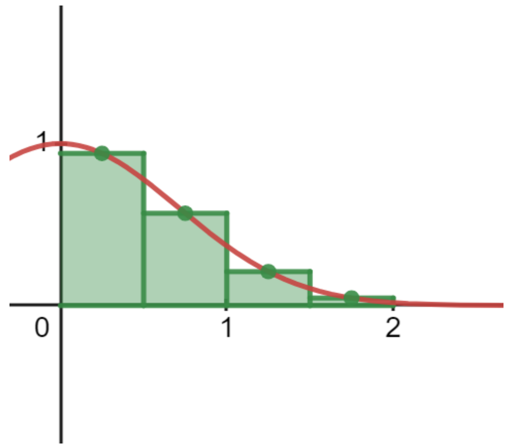
The Mid-Ordinate Rule

The mid-ordinate rule is a way of approximating the area under a curve by using vertical strips of equal width. Instead of using the endpoints of each vertical strip, the y-value of the curve at each midpoint is used to fit a rectangle. The area of each rectangle is found by multiplying the width of the strip with the height at the midpoint, and then all are summed to give the final area.



Using n equally sized intervals with end points $a = x_0, x_1, \dots, x_n = b$, the mid-ordinate rule states that:

$$\int_a^b f(x)dx \approx h\left[y_{\frac{1}{2}} + y_{\frac{3}{2}} + \cdots y_{n-\frac{1}{2}}\right]$$

Where $h = \frac{b-a}{n}$. This will be given in the formula book.

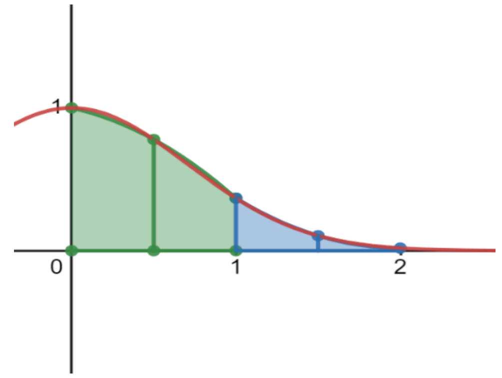
Example 1: Using the mid-ordinate rule with four strips, approximate $\int_0^2 e^{-x^2}dx$ to three significant figures (3 s.f.)

	Strip endpoint	Strip midpoint	y-value of midpoint
First, divide the range from 0 to 2 into 4 strips. As with other numerical method techniques, it may be easier to lay out the working in a table. Calculate the midpoint and the corresponding y-value for the midpoint in each strip. As this is to be given to 3 s. f., make sure to note to at least four in order to preserve a high enough level of accuracy before rounding.	$x_0 = 0$		
		$x_{0.5} = 0.25$	$y_{0.5} = 0.9394$
	$x_1 = 0.5$		
		$x_{1.5} = 0.75$	$y_{1.5} = 0.5698$
	$x_2 = 1$		
		$x_{2.5} = 1.25$	$y_{2.5} = 0.2096$
	$x_3 = 1.5$		
		$x_{3.5} = 1.75$	$y_{3.5} = 0.0468$
	$x_4 = 2$		
Using the formula, substitute the values in and round to the appropriate number of significant figures.	$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5.$ $\int_0^2 e^{-x^2}dx \approx 0.5[0.9394 + 0.5698 + 0.2096 + 0.0468]$ $= 0.883$ to 3 s.f.		

For convex curves, the mid-ordinate rule gives an underestimate. For concave curves, the rule gives an overestimate. The error for the mid-ordinate rule will be in the opposite direction as that for the trapezium rule and give approximately half the magnitude of error.

Simpson's Rule

Instead of using a straight line to approximate points on a curve, you can fit a quadratic to the curve. To define a quadratic, three points are needed, and so using the endpoints of two adjacent strips allows that section of the curve to be approximated.



For two adjacent strips with coordinates $(x_0, y_0); (x_1, y_1)$ and $(x_1, y_1); (x_2, y_2)$, it can be shown that the area under the quadratic fitted to those points is equal to: $\frac{d}{3}(y_0 + 4y_1 + y_2)$. Then, Simpson's rule states that for an even number n of equally sized intervals over a range $a = x_0, x_1, \dots, x_n = b$, the integral of the curve can be approximated by:

$$\int_a^b f(x)dx \approx \frac{h}{3}[y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})]$$

Where $h = \frac{b-a}{n}$. This will be given in the formula book.

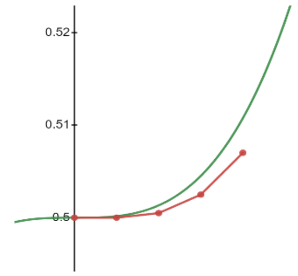
Example 2: Using Simpson's rule with four strips, approximate $\int_0^2 e^{-x^2}dx$ to three significant figures.

	x	$y = \exp(-x^2)$	Multiple
First, divide the range from 0 to 2 into 4 strips. As with other numerical method techniques, it may be easier to lay out the working in a table.	$x_0 = 0$	$y_0 = 1$	1
	$x_1 = 0.5$	$y_1 = 0.7788$	4
	$x_2 = 1$	$y_2 = 0.3679$	2
	$x_3 = 1.5$	$y_3 = 0.1054$	4
	$x_4 = 2$	$y_4 = 0.0183$	1
Using Simpson's rule, we see that the end points will be summed once, the even-numbered points will be included twice, and the odd numbered points will be included four times. We now have the value of the sum in the square brackets of Simpson's rule.	$y_0 + y_n + 2(y_2 + y_4 + \cdots + y_{n-2}) + 4(y_1 + y_3 + \cdots + y_{n-1})$ $= y_0 + y_4 + 2(y_2) + 4(y_1 + y_3)$ $= 1 + 0.0183 + 2(0.3679) + 4(0.7788 + 0.1054)$ $= 5.2909$		
Now that the inner sum is complete, we calculate the value for h and substitute it into the formula to give the final value.	$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$ $\int_0^2 e^{-x^2}dx \approx \frac{0.5}{3}[5.2909] = 0.8818$ $= 0.882$ to 3 s.f.		

The true value of the integral given in these two examples is 0.88208 (to 5 s.f.). Simpson's rule provides a slightly better approximation for the exact same number of strips - this is generally how Simpson's rule compares to linear approximations.

Euler's Method

Unlike the other methods in this section, Euler's method is not used for the approximation of definite integrals. Instead, it is a method that uses a function of the derivative – a differential equation – along with a known point to approximate another point's position by extending the tangent. This method uses principles relating to the Maclaurin expansion of functions.



Recall that the Maclaurin series of a function $f(x)$ is given by:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots$$

Translating this expansion so that it is centred around a value a gives the Taylor series:

$$f(a + x) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \cdots$$

A simple approximation of a curve by its Taylor series uses the first two terms of the sum:

$$f(a + h) \approx f(a) + hf'(a).$$

If the x value is far away from the known point, the approximation becomes more unreliable. To combat this, an iterative method can be used in order to recalculate the gradient each time you move along the x-axis in a small step, h: this is Euler's Method.

For $\frac{dy}{dx} = f(x, y)$ and a known point $y(x_0) = y_0$,

$$y_{r+1} \approx y_r + hf(x_r, y_r) \text{ where } x_{r+1} = x_r + h.$$

The smaller the value of h, the more accurate the approximation will be.

Example 3: For the differential equation $\frac{dy}{dx} = x^2y$ with known value $y(0) = 0.5$, use Euler's method to find an approximation of y when $x = 0.4$, to 3 d.p.

First, state the start value and the interval value, h.	$x_0 = 0, y_0 = 0.5, h = 0.1$
At each step, apply the formula for Euler's method: $y_{r+1} = y_r + hf(x_r, y_r)$.	$y_1 = 0.5 + 0.1(0^2 \times 0.5) = 0.5$ $x_1 = 0 + 0.1 = 0.1$ $y_2 = 0.5 + 0.1(0.1^2 \times 0.5) = 0.5005$ $x_2 = 0.1 + 0.1 = 0.2$ $y_3 = 0.502502 + 0.1(0.2^2 \times 0.5005) = 0.502502$ $x_3 = 0.2 + 0.1 = 0.3$ $y_4 = 0.502502 + 0.1(0.3^2 \times 0.5005) = 0.507024518$ $x_4 = 0.3 + 0.1 = 0.4$ $y_4 = 0.507$ to 3.d.p.