Matrices IV Cheat Sheet

Factorisation of Determinants Using Row and Column Operations (A Level Only)

A triangular matrix is a matrix which has non-zero entries only on the leading diagonal (top left to bottom right) and either above or below the leading diagonal.

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} \qquad \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$$

Upper Triangular Matrix — Lower Triangular Matrix

The determinant of a triangular matrix can be found by multiplying all entries on the leading diagonal line.

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = a\begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - 0\begin{vmatrix} b & c \\ 0 & f \end{vmatrix} + 0\begin{vmatrix} b & c \\ d & e \end{vmatrix} = adf$$

Row and column operations can be used to manipulate a matrix into either of these forms so that the calculations for their determinants can be simplified. The table below contains some row and column operations and the rules that apply to their determinants, for when matrix B is derived from matrix A using row or column operations.

Operation	Determinant
Swapping between two rows or two columns	$\det B = -\det A$
Multiplying every entry in the matrix by a constant k	$\det B = k^3 \det A$
Multiplying either every entry in one row or one column with a constant k	$\det B = k \det A$
Adding a multiple of one row to corresponding entries in another row, or from one column to another	$\det B = \det A$

Example 1: Without expanding, find the determinant of the matrix $\begin{bmatrix} x+1 & 7 & 3 \\ 2x+2 & 2x+8 & 10 \\ x+1 & 7 & x+5 \end{bmatrix}$ in terms of x.

Notice that in the first two columns, subtracting R_1 from R_3 gives two terms of 0 in R_3.	$\begin{vmatrix} x+1 & 7 & 3 \\ 2x+2 & 2x+8 & 10 \\ x+1 & 7 & x+5 \end{vmatrix}$ $= \begin{vmatrix} x+1 & 7 & 3 \\ 2x+2 & 2x+8 & 10 \\ 0 & 0 & x+2 \end{vmatrix}$
Notice that all entries in R_2 are multiples of 2, so 2 can be factorised out of the determinant.	$= \begin{vmatrix} 2x+1 & 7 & 3 \\ 2(x+1) & 2(x+4) & 2(5) \\ 0 & 0 & x+2 \end{vmatrix}$ $= 2\begin{vmatrix} x+1 & 7 & 3 \\ x+1 & x+4 & 5 \\ 0 & 0 & x+2 \end{vmatrix}$
Factorise $(x+1)$ from C_1.	$= 2(x+1)\begin{vmatrix} 1 & 7 & 3 \\ 1 & x+4 & 5 \\ 0 & 0 & x+2 \end{vmatrix}$
Subtract R_1 from R_2 to get a triangular matrix.	$= 2(x+1)\begin{vmatrix} 1 & 7 & 3 \\ 0 & x-3 & 2 \\ 0 & 0 & x+2 \end{vmatrix}$
Multiply the entries of the leading diagonal with $2(x+1)$.	$2(x+1)(x-3)(x+2)$

In the following situations, the determinant of the matrix is 0:

- When a row is the same as another row, or a column is the same as another column
- When a row is a multiple of another row, or a column is a multiple of another column
- When a row is a linear combination of the other 2 rows, or a column is a linear combination of the other 2 columns

Eigenvalues and Eigenvectors (A Level Only)

Eigenvectors show the direction of all invariant lines passing through the origin for a given transformation. Eigenvalues show the scale factor by which the transformation enlarges the eigenvector. For a square matrix M,

$$Mu = \lambda u$$

Where u is its eigenvector and λ is the corresponding scalar eigenvalue. Note that u is a non-zero vector.

The characteristic equation can be derived from the equation above, using the identity matrix.

$$Mu - \lambda u = 0$$

$$\Rightarrow (M - \lambda I)u = 0$$

For the equation to be equal to zero when u is a non-zero vector, $\det(M - \lambda I)$ must be 0. Hence, the characteristic equation is written as below. The eigenvalues are the solutions to this.

$$\det(M - \lambda I) = 0$$

Example 2: Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 2 & 2 \\ 10 & 3 \end{bmatrix}$.

Find the eigenvalues using the characteristic equation $\det(M - \lambda I) = 0$.	$M - \lambda I = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $= \begin{bmatrix} 2-\lambda & 4 \\ 5 & 3-\lambda \end{bmatrix}$ $\det(M - \lambda I) = (2-\lambda)(3-\lambda) - (4)(5)$ $= 6 - 3\lambda - 2\lambda + \lambda^2 - 20$ $= \lambda^2 - 5\lambda - 14$ $\det(M - \lambda I) = 0$ $\Rightarrow \lambda^2 - 5\lambda - 14 = 0$ $\Rightarrow (\lambda+2)(\lambda-7) = 0$ $\Rightarrow \lambda = -2, \lambda = 7$
Using $(M - \lambda I)u = 0$ and $u = \begin{bmatrix} x \\ y \end{bmatrix}$, solve for the corresponding eigenvectors for each eigenvalue found by using simultaneous equations.	$\lambda = -2,$ $\begin{bmatrix} 4 & 4 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $4x + 4y = 0$ $x = -y \Rightarrow u = \begin{bmatrix} k \\ -k \end{bmatrix}$ $\therefore u = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ where k is any non-zero scalar multiple of the vector. $\lambda = 7,$ $\begin{bmatrix} -5 & 4 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $-5x + 4y = 0$ $5x = 4y \Rightarrow u = \begin{bmatrix} 4k \\ 5k \end{bmatrix}$ $\therefore u = k\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ where k is any non-zero scalar multiple of the vector.

The eigenvalues and eigenvectors for a 3×3 matrix can also be found in the same way.

Reducing Matrices to a Diagonal Form

Diagonalisation of a General Square Matrix

A diagonal matrix is a square matrix which only has non-zero values on its leading diagonal. For a matrix M with the eigenvectors u_1, u_2 and u_3, the matrix U can be constructed by writing each eigenvector as a column of the matrix.

$$U = [u_1 \quad u_2 \quad u_3]$$

Using $Mu = \lambda u$ and $\lambda_1, \lambda_2, \lambda_3$ as the corresponding eigenvalues for the eigenvectors u_1, u_2 and u_3,

$$MU = M[u_1 \quad u_2 \quad u_3]$$

$$= [\lambda_1 u_1 \quad \lambda_2 u_2 \quad \lambda_3 u_3]$$

$$= [u_1 \quad u_2 \quad u_3]\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$= UD$$

where D is a diagonal matrix containing all the respective eigenvalues in its leading diagonal.

Given that U is always non-singular,

$$M = UDU^{-1}$$

The same approach can be used to diagonalise a 2×2 matrix as well. The resulting diagonal matrix will be in the form of $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

Raising a Matrix to a Power

Diagonalisation of square matrices allows the calculation to be simplified when raising a matrix to a power. For any positive or negative integers of k, when M is non-singular,

$$M^k = UD^kU^{-1}$$

For the diagonal matrix D, D^k can be found straight away by raising the power of each term to the power of k.

Example 3: Given that the matrix M has eigenvalues $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = -1$ and the eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, find M^3.

Construct U using each eigenvector as a column in the matrix.	$U = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & -1 & 4 \end{bmatrix}$
Construct D using the eigenvalues as the leading diagonal.	$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Find D^3 by raising the power of each term to 3.	$D^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Find U^{-1} using a calculator.	$U^{-1} = \begin{bmatrix} 10 & -13 & 4 \\ -2 & 3 & -1 \\ -3 & 4 & -1 \end{bmatrix}$
Find M^3 using $M^k = UD^kU^{-1}$.	$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 10 & -13 & 4 \\ -2 & 3 & -1 \\ -3 & 4 & -1 \end{bmatrix}$ $= \begin{bmatrix} 0 & 24 & -1 \\ 0 & 16 & -2 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} 10 & -13 & 4 \\ -2 & 3 & -1 \\ -3 & 4 & -1 \end{bmatrix}$ $= \begin{bmatrix} -48+3 & 72-4 & -24+1 \\ -32+6 & 48-8 & -24+1 \\ 16+12 & -24-16 & 8+4 \end{bmatrix}$ $= \begin{bmatrix} -45 & 68 & -23 \\ -26 & 40 & -14 \\ 28 & -40 & 12 \end{bmatrix}$