Matrices III Cheat Sheet (A-Level Only)

Solving Systems of Three Simultaneous Equations

Three simultaneous equations containing three unknowns can be rewritten and solved using matrices.

$$ax + by + cz = p$$
$$dx + ey + fz = q$$
$$gx + hy + iz = r$$

The above equations can be rewritten as:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

When $\det A = 0$, the coefficient matrix A is a singular matrix and there is either no solution or infinitely many solutions. When $\det A \neq 0$, the matrix is non-singular and there is a unique solution to the equations.

For a non-singular matrix of coefficients, A:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Example 1: For the following system of simultaneous equations, **a.)** express it as a matrix equation; **b.)** show whether there is a unique solution; **c.)** find the values of x, y and z.

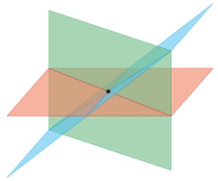
$$4x + y + 2z = 16$$
$$3x + 4y - 2z = 24$$
$$-x + y + z = 7$$

a.) Rewrite the equations as matrices.	$A = \begin{bmatrix} 4 & 1 & 2 \\ 3 & 4 & -2 \\ -1 & 1 & 1 \end{bmatrix}$ $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 24 \\ 7 \end{bmatrix}$
b.) Find $\det A$ using a calculator.	$\det A = 37$ $\det A \neq 0$, so there is a unique solution.
c.) Find A^{-1} using a calculator.	$A^{-1} = \frac{1}{37}\begin{bmatrix} 6 & 1 & -10 \\ -1 & 6 & 14 \\ 7 & -5 & 13 \end{bmatrix}$
Solve for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 16 \\ 24 \\ 7 \end{bmatrix}$ $= \frac{1}{37}\begin{bmatrix} 6 & 1 & -10 \\ -1 & 6 & 14 \\ 7 & -5 & 13 \end{bmatrix}\begin{bmatrix} 16 \\ 24 \\ 7 \end{bmatrix}$ $= \frac{1}{37}\begin{bmatrix} 6(16) + 24 - 10(7) \\ -16 + 6(24) + 14(7) \\ 7(16) - 5(24) + 13(7) \end{bmatrix}$ $= \frac{1}{37}\begin{bmatrix} 50 \\ 226 \\ 83 \end{bmatrix}$
Write down the values of x, y and z.	$x = \frac{50}{37}, y = \frac{226}{37}, z = \frac{83}{37}$

Geometric Meaning of the Solution of Three Simultaneous Equations

A set of three simultaneous equations with three variables describes the arrangement of three planes in 3D space.

Single Point



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The three planes intersect at a single point, so there is a unique solution. This is the only case where the coefficient matrix A is non-singular. The system of equations is consistent.

Sheaf and Coincident Planes

The planes intersect at either a line or a plane, so there are infinitely many solutions. The coefficient matrix A is singular and the system of equations is consistent.

a) When all three equations describe the same plane, the solution is the plane itself.

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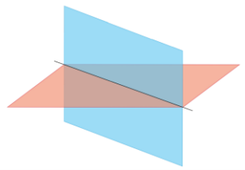
b) When two of the equations describe the same plane and the third plane is not parallel to this plane, the planes will intersect along a line.



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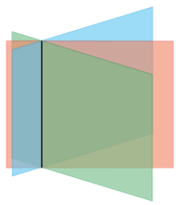
c) When all three equations describe different planes, but they all intersect along a line, they form a sheaf.



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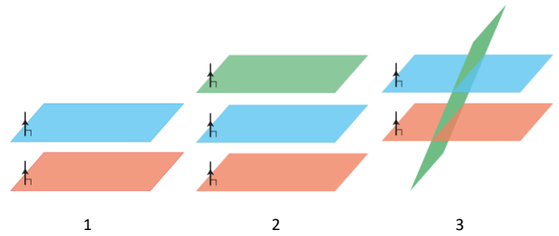
Triangular Prism and Parallel Planes

The coefficient matrix A is singular and the three planes do not intersect, so there are no unique solutions.

a) When there are two distinct parallel planes, there are no solutions. The third plane can be (1) the same as one of the other two planes, (2) parallel to both planes, or (3) cut through both planes. It can intersect the planes along different lines. The system of equations is inconsistent.



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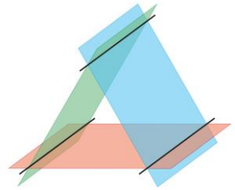
b) A triangular prism is where each pair of planes intersects along a straight line. The three lines of intersection are parallel to each other.



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Example 2: For the following system of equations

$$x - 3y - 3z = 8$$
$$2x - y + z = 6$$
$$3x + y + 5z = -7$$

Show that **a.)** there is no unique solution; **b.)** the system is inconsistent. **c.)** Interpret the geometric meaning of this system.

a.) Find the determinant of the matrix of coefficients.	$A = \begin{bmatrix} 1 & -3 & -3 \\ 2 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix}$ $\det A = 0$ (using calculator), so the matrix is singular and there is no unique solution.
b.) Eliminate x from the equations.	Equation ①: $x - 3y - 3z = 8$ Equation ②: $2x - y + z = 6$ Equation ③: $3x + y + 5z = -7$ Equation ④: ①× 2: $2x - 6y - 6z = 16$ Equation ⑤: ②−④: $5y + 7z = -10$ Equation ⑥: ①× 3: $3x - 9y - 9z = 24$ Equation ⑦: ③−⑥: $10y + 14z = -31$
Compare the 2 equations without the variable x to show that the system is inconsistent.	Equation ⑤: $5y + 7z = -10$ Equation ⑦: $10y + 14z = -31$ ⑤× 2: $10y + 14z = -20$ $-20 \neq -31$, so the system is inconsistent.
c.) Interpret the geometric meaning.	The system is inconsistent, and no rows are multiples of other rows, so there are no parallel planes. The three planes form a triangular prism.