Matrices Cheat Sheet II

Determinant of a Matrix

The determinant of a matrix is a scalar value which can be calculated from a square matrix. For a matrix A, the determinant can be denoted by $\det(A)$, $\det A$, $|A|$ or Δ.

2×2 Matrices

For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$|A| = ad - bc$$

The determinant of a matrix represents the area scale factor of the transformation. The area of the image can be found by multiplying the area of the object by the determinant. A determinant of 0 shows that the transformation maps to an image with an area of 0. This means that the transformation maps all points to a straight line, except in a special case where the zero matrix maps all points to the origin. If the determinant is negative, it shows that the order of vertices has been reversed.

Example 1: An object with an area of 4cm^2 is transformed by the matrix $\begin{bmatrix} 4 & 3 \\ 10 & 6 \end{bmatrix}$. Find the area of the image after transformation.

Find the determinant from the matrix.	$ad - bc = (4 \times 6) - (3 \times 10)$ $= 24 - 30$ $= -6$
Multiply the determinant with the original area.	$-6 \times 4 = -24$
An area cannot be negative so ignore the negative sign.	24cm^2

3×3 Matrices (A Level Only)

The determinant for a 3×3 matrix can also be denoted as $|a\ b\ c|$. For a matrix $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$,

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

- $\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}$ is known as the submatrix of a_1 and is obtained from deleting the row and column which a_1 is in.
- The minor of a_1 is the determinant of its submatrix, which is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$.
- The cofactor of a_1 is given by its minor multiplied by $(-1)^{i+j}$, where i and j are the row and column numbers which a_1 is in. Generally, a cofactor is a minor with the corresponding sign: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$
- The formula above is the sum of the product of values in the first column with their cofactors.
- The same determinant can be obtained by expanding any row or column using the same method.

The determinant of a 3×3 matrix is also its volume scale factor. A negative determinant shows that the orientation of the object is reversed after transformation.

Example 2: Find the determinant of $\begin{bmatrix} 1 & 8 & 1 \\ 5 & 1 & 4 \\ 2 & 7 & 2 \end{bmatrix}$. Hence comment on the orientation and volume of the object after transformation.

Find the determinant from the matrix.	$|a\ b\ c| = 1\begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} - 5\begin{vmatrix} 8 & 1 \\ 7 & 2 \end{vmatrix} + 2\begin{vmatrix} 8 & 1 \\ 1 & 4 \end{vmatrix}$ $= 1(1 \times 2 - 4 \times 7) - 5(8 \times 2 - 1 \times 7) + 2(8 \times 4 - 1 \times 1)$ $= (-26) - 5(9) + 2(31)$ $= -9$
The sign represents the orientation of the image and the value represents the volume scale factor.	The image will have a volume which is 9 times the original volume. The negative sign shows that the orientation is reversed.

Inverse of a Matrix

An inverse matrix is similar to the reciprocal used for scalars and is used to reverse matrix multiplication. For a square matrix A, its inverse is denoted by A^{-1}. When a matrix is multiplied by its inverse, no matter the order, the product is always the identity matrix, I.

$$AA^{-1} = A^{-1}A = I$$

Not all matrices have an inverse. These are called singular matrices and have a determinant of 0.

Example 3: Given that A is not a singular matrix and $AX = B$, find X.

Pre-multiply both sides of the equation by A^{-1}.	$A^{-1}AX = A^{-1}B$
Substitute $A^{-1}A = I$ into the equation.	$IX = A^{-1}B$
Substitute $IX = X$ into the equation.	$X = A^{-1}B$

For $XA = B$, $X = BA^{-1}$. Notice that the order of multiplying is important here.

For square matrices of the same size,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of Non-singular 2×2 Matrices

For a non-singular 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse can be found using the following:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of a transformation matrix can be used to find the coordinates of the object, when the coordinates of the image are given.

Example 4: Point X is mapped onto point $Y(7,2)$ under the transformation $A = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix}$. Find the coordinates of point X.

Find the determinant of A.	$\det A = (6 \times 5) - (1 \times 2)$ $= 28$
Find the inverse of A.	$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $= \frac{1}{28}\begin{bmatrix} 5 & -1 \\ -2 & 6 \end{bmatrix}$
Since $AX = Y$, it follows that $X = A^{-1}Y$.	$A^{-1}\begin{bmatrix} 7 \\ 2 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 5 & -1 \\ -2 & 6 \end{bmatrix} \times \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ $= \frac{1}{28}\begin{bmatrix} (5 \times 7) + (-1 \times 2) \\ (-2 \times 7) + (6 \times 2) \end{bmatrix}$ $= \frac{1}{28}\begin{bmatrix} 33 \\ -2 \end{bmatrix}$ $= \begin{bmatrix} \frac{33}{28} \\ -\frac{1}{14} \end{bmatrix}$
Write down the coordinates of X.	$X\left(\frac{33}{28}, -\frac{1}{14}\right)$

It is possible to find unknown entries in a matrix when the determinant is given.

Example 5: Given that a matrix $\begin{bmatrix} x & -8 \\ 4 & x+1 \end{bmatrix}$ has a determinant of 38, find the 2 possible matrices.

Find the determinant of the matrix in terms of x.	$ad - bc = (x)(x+1) - (-8 \times 4)$ $= x^2 + x + 32$
Equate that to the given determinant.	$x^2 + x + 32 = 38$
Solve for possible values of x.	$x^2 + x - 6 = 0$ $(x+3)(x-2) = 0$ $x = -3$ or $x = 2$
Find the possible solutions.	$\begin{bmatrix} -3 & -8 \\ 4 & -2 \end{bmatrix}$ and $\begin{bmatrix} 2 & -8 \\ 4 & 3 \end{bmatrix}$

Inverse of 3×3 Matrices (A Level Only)

Transposition is when the rows and columns within a matrix are swapped. The transpose of a matrix A is denoted by A^T. For matrices A and B which have the same size,

$$(AB)^T = B^T A^T$$

$$(A + B)^T = A^T + B^T$$

The inverse of a 3×3 non-singular matrix A is given by:

$$A^{-1} = \frac{1}{\det A} C^T$$

- C represents the cofactor matrix, in which each element in the matrix is represented by its cofactor.
- C^T represents that the cofactor matrix is transposed.

Example 6: Find the inverse of the matrix $\begin{bmatrix} 1 & 8 & 1 \\ 5 & 1 & 4 \\ 2 & 7 & 2 \end{bmatrix}$.

Find the cofactor of each element.	$C = \begin{bmatrix} \begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 5 & 1 \\ 2 & 7 \end{vmatrix} \\ -\begin{vmatrix} 8 & 1 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 8 \\ 2 & 7 \end{vmatrix} \\ \begin{vmatrix} 8 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 8 \\ 5 & 1 \end{vmatrix} \end{bmatrix}$ $= \begin{bmatrix} -26 & -2 & 33 \\ -9 & 0 & 9 \\ 31 & 1 & -39 \end{bmatrix}$
Transpose C.	$C^T = \begin{bmatrix} -26 & -9 & 31 \\ -2 & 0 & 1 \\ 33 & 9 & -39 \end{bmatrix}$
Find the determinant. (Given in example 2)	-9
Find the inverse matrix.	$\frac{1}{\det A}C^T = -\frac{1}{9}\begin{bmatrix} -26 & -9 & 31 \\ -2 & 0 & 1 \\ 33 & 9 & -39 \end{bmatrix}$ $= \begin{bmatrix} \frac{26}{9} & 1 & -\frac{31}{9} \\ \frac{2}{9} & 0 & -\frac{1}{9} \\ -\frac{11}{3} & -1 & \frac{13}{3} \end{bmatrix}$