Matrices I Cheat Sheet

Matrix Arithmetic and Multiplying a Matrix by a Scalar

A matrix is an array of numbers, or elements, arranged in rows and columns. Below are some examples of matrices:

$$A = \begin{bmatrix} 7 & 4 & 2 \\ 4 & 9 & 13 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 4 & 8 \\ 10 & 5 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 2 \\ 7 & 4 \\ 6 & 3 \end{bmatrix}$$

The order of a matrix is described by the number of rows and columns in the form $r \times c$. In the above examples, the order of A and B are 2×3 and the order of $C = 3 \times 2$.

Matrices with the same order can be added together or subtracted from one another, by adding or subtracting elements in the corresponding positions. These are known as conformable matrices.

Example 1: Evaluate the following matrix sum: $\begin{bmatrix} 7 & 4 \\ 4 & 9 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 10 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 6 & 11 \end{bmatrix}$

Check if all matrices have the same order.	All matrices are 2×2 so they are conformable.
Add numbers in corresponding position e.g. $7 + 6 = 13$.	$\begin{bmatrix} 13 & 8 \\ 14 & 14 \end{bmatrix}$
Subtract numbers in corresponding positions.	$\begin{bmatrix} 11 & 8 \\ 8 & 3 \end{bmatrix}$

The orders of matrices show whether the matrices are conformable for multiplication and what matrix order the final product will have. The number of columns in the first matrix must be equal to the number of rows in the second. For example:

$$(2 \times 1) \times (1 \times 3)$$

- The middle numbers are the same, showing that the matrices are conformable for multiplication.
- The first and last numbers show that the final product will have an order of 2×3.

Example 2: Evaluate the following matrix multiplication: $\begin{bmatrix} 6 & 4 & 8 \\ 10 & 5 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ 7 & 4 \\ 6 & 3 \end{bmatrix}$

Check the order of matrices.	$2 \times 3 \quad \times \quad 3 \times 2$ Matrices are conformable and final matrix has order of 2×2
Multiply the numbers of the first row of the first matrix with the corresponding numbers of the first column of the second matrix and add them up, this will be the value in the first row of the first column in the final answer.	$6 \times 4 + 4 \times 7 + 8 \times 6 = 100$
Multiply the numbers of the first row of the first matrix with the corresponding numbers of the second column of the second matrix and add them up, this will be the value in the first row of the second column in the final answer.	$6 \times 2 + 4 \times 4 + 8 \times 3 = 52$
Repeat for the second row of the first matrix.	$10 \times 4 + 5 \times 7 + 1 \times 6 = 81$ $10 \times 2 + 5 \times 4 + 1 \times 3 = 43$
Final answer.	$\begin{bmatrix} 100 & 52 \\ 81 & 43 \end{bmatrix}$

Matrices can also be multiplied by a scalar number. In that case, just multiply each element of the matrix by the scalar number.

Example 3: Evaluate the following: $4\begin{bmatrix} 4 & 2 \\ 7 & 4 \\ 6 & 3 \end{bmatrix}$

Multiply each number in the matrix by 4.	$\begin{bmatrix} 4 \times 4 & 4 \times 2 \\ 4 \times 7 & 4 \times 4 \\ 4 \times 6 & 4 \times 3 \end{bmatrix}$
Final answer.	$\begin{bmatrix} 16 & 8 \\ 28 & 16 \\ 24 & 12 \end{bmatrix}$

Zero and Identity Matrices

In zero matrices, all elements are zero. They can come in any order. Below are all examples of zero matrices:

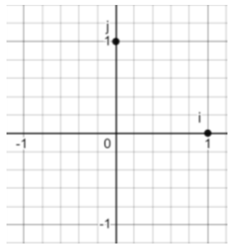
$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity matrices are often denoted by I and are always square matrices. The diagonal elements from top left to bottom right are always ones, and all other elements are always zeros. When a matrix is multiplied with the identity matrix of the same order, the product should equal itself. Here are two examples of identity matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrices as Transformations

The transformation of an object into an image can be represented using matrices. The transformation matrix can be found by mapping the transformation of two unit vectors, $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The resulting position vectors form the transformation matrix. For example, when reflected in the x axis, the images of i and j have the position vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Thus, the matrix for reflection in the x axis is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.



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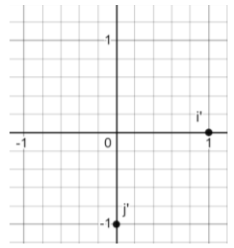

Image

The table below summarises some common transformation vectors and what they mean.

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	Reflection in the line $y = x$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$180°$ rotation about the origin
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	Reflection in the x axis	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	$90°$ rotation clockwise about the origin
$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$	Reflection in the y axis	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$270°$ rotation clockwise about the origin
$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$	Reflection in the line $y = (\tan\theta)x$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\theta°$ rotation anticlockwise about the origin
$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$	Stretch with scale factor k parallel to x axis	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$	Shear with x axis fixed
$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$	Stretch with scale factor k parallel to y axis	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$	Shear with y axis fixed
$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	Enlargement with scale factor k with origin as centre		

When an object undergoes two successive transformations S and T, a single transformation matrix, $T \times S$ can be found. Note that the matrices are multiplied in reverse order.

Example 4: A triangle with vertices at the coordinates $A(1,3)$, $B(1,7)$ and $C(2,6)$ is reflected in the x axis and then rotated $45°$ anticlockwise about the centre of the origin. Find the coordinates of its image.

Find the transformation matrix for reflection in the x axis.	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Find the transformation matrix for $45°$ anticlockwise rotation about the centre of origin.	$\begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
Multiply the transformation matrices together in reverse order.	$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix}$
Multiply the combined transformation matrix with the position vector of the object.	$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}}{2} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 2 \\ 3 & 7 & 6 \end{bmatrix}$ $= \begin{bmatrix} 2\sqrt{2} & 4\sqrt{2} & 4\sqrt{2} \\ -\sqrt{2} & -3\sqrt{2} & -2\sqrt{2} \end{bmatrix}$
Write down the coordinates of each vector.	$A(2\sqrt{2}, -\sqrt{2}), B(4\sqrt{2}, -3\sqrt{2}), C(4\sqrt{2}, -\sqrt{2})$

In 3 dimensions, transformation matrices are 3×3 and can be found in a similar way as in 2 dimensions, using the unit vectors $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	Reflection in the xy plane or $z = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	Rotation of $\theta°$ about x axis
$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Reflection in the xz plane or $y = 0$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	Rotation of $\theta°$ about y axis
$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Reflection in the yz plane or $x = 0$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	Rotation of $\theta°$ about z axis

Invariant Points and Lines of a Linear Transformation

A point is invariant if its image after transformation is mapped onto itself. A line of invariant points is a line which all points map onto themselves. On the other hand, an invariant line consists of points which are mapped onto any point of the line - not necessarily the object point.

Example 5: Show that $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is an invariant point for the transformation $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. Find the line of invariant for this transformation.

Multiply $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ to find the image.	$\begin{bmatrix} 0 + 3 \\ -3 + 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ The object and image have the same matrices, so it is an invariant point.
Use the conditions of invariant points to form two simultaneous equations.	$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ $0x + y = x$ $-x + 2y = y$
Simplify the equations to find the line of invariant points.	$y = x$

Example 6: Find the invariant line for the transformation $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$.

Equate $\begin{bmatrix} 3 & 4 \\ 9 & -2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$	$x' = 3x + 4y$ $y' = 9x - 2y$
Substitute $y = mx + c$ into both equations.	$x' = 3x + 4mx + 4c$ $y' = 9x - 2mx - 2c$
Equate $y' = mx' + c$.	$9x - 2mx - 2c = 3mx + 4m^2x + 4mc$ $0 = 4m^2x + 5mx - 9x + 4mc + 2c$ $= (4m^2 + 5m - 9)x + (4m + 2)c$
Solve for values of m and c for LHS to become 0.	$(4m + 9)(m - 1) = 0$ or $4m + 2 = 0$ $m = \frac{-9}{4}, m = 1$ Note: $m = -\frac{1}{2}$ is not applicable because $4m^2 + 5m - 9 \neq 0$
To satisfy the equation, $(4m + 2)c$ must also be 0.	$c = 0$
Invariant lines.	$y = -\frac{9}{4}x \qquad y = x$