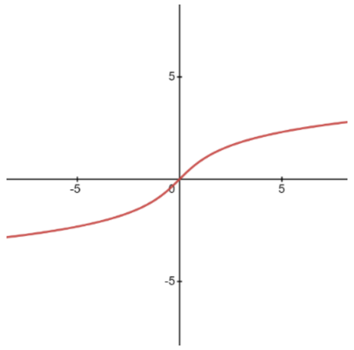
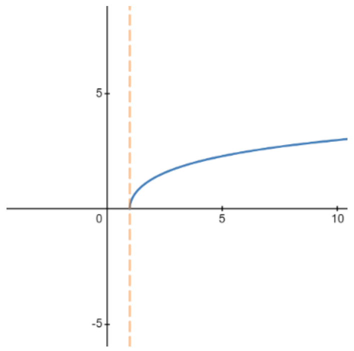
Inverse Hyperbolic Functions

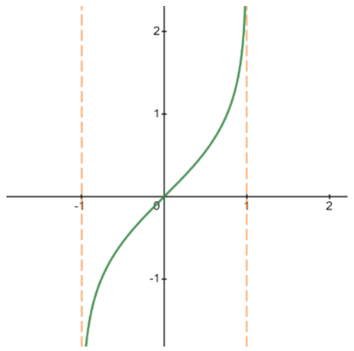
As with their trigonometric counterparts, $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ all have inverse functions.



$\sinh^{-1}(x) = \text{arsinh}(x)$



$\cosh^{-1}(x) = \text{arcosh}(x)$



$\tanh^{-1}(x) = \text{artanh}(x)$

As with all inverse functions, the graphs are reflections of the function in the line y = x.

Logarithmic form of Inverse Hyperbolic Functions

The exact forms of the inverse hyperbolic functions are given below:

$$\text{arsinh}(x) = \ln\left(x + \sqrt{x^2+1}\right) \qquad \text{arcosh}(x) = \ln\left(x + \sqrt{x^2-1}\right) \qquad \text{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

These will also be given in your formula booklet. You are expected to be able to prove these results using the exponential definitions of hyperbolic functions.

Example 1: Prove that $\text{arsinh}(x) = \ln(x + \sqrt{x^2+1})$.

First, let $y = \text{arsinh}(x)$ with the aim to rearrange to find a new expression for y in terms of x.	$y = \text{arsinh}(x)$
This is found by taking the $\sinh(x)$ of both sides.	This then means that $\sinh(y) = x$.
Using the exponential form of $\sinh(x)$.	$\frac{e^y - e^{-y}}{2} = x$
Rearrange to find a disguised quadratic in terms of e^y.	$e^y - e^{-y} = 2x$ $e^y - \frac{1}{e^y} = 2x$ $(e^y)^2 - 1 = 2xe^y$ $(e^y)^2 - 2xe^y - 1 = 0$
Use the quadratic formula to find a solution for e^y.	$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$ $e^y = \frac{2x \pm \sqrt{4(x^2+1)}}{2}$ $e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$ $e^y = x \pm \sqrt{x^2+1}$
As $\text{arsinh}(x)$ is a function, there can only be one value of y for each value of x. Conventionally we take the positive root. This makes $e^y > 1$ and $y > 0$. Take logs of both sides.	$e^y = x + \sqrt{x^2+1}$ $y = \ln(x + \sqrt{x^2+1})$
As $y = \text{arsinh}(x)$, this means that:	$\text{arsinh}(x) = \ln(x + \sqrt{x^2+1})$

The same method can be used to prove the result for $\text{arcosh}(x)$.

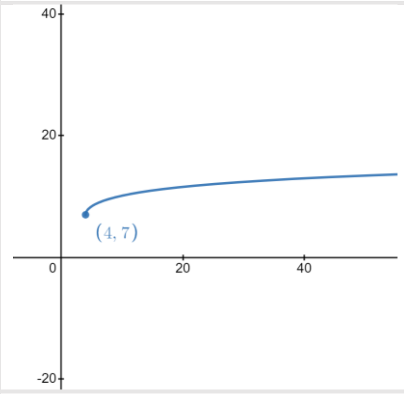
Domains and Ranges of Inverse Hyperbolic Functions (A Level Only)

The domains and ranges of the inverse hyperbolic functions are given in the table below. Note that the range of a function is the domain of its inverse.

Function	Domain	Range
$\text{arsinh}(x)$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
$\text{arcosh}(x)$	$x \geq 1$	$f(x) \geq 0$
$\text{artanh}(x)$	$-1 < x < 1$	$f(x) \in \mathbb{R}$

Example 3: Let $f(x) = 7 + 2\,\text{arcosh}\left(\frac{x}{4}\right)$.

a) State the largest possible domain of $f(x)$.
b) For the domain in part a) find the range of $f(x)$.

a) Using the domain of $\text{arcosh}(x)$ is $x \geq 1$, redefine the domain for when $\frac{x}{4} \geq 1$.	$\frac{x}{4} \geq 1$ then $x \geq 4$.
b) When determining the range of a function, it is always useful to sketch the graph. The transformations required for the transformation from $\text{arcosh}(x)$ to $f(x) = 7 + 2\,\text{arcosh}\left(\frac{x}{4}\right)$ are: - stretch of scale factor 4 parallel to the x axis - stretch of scale factor 2 parallel to the y axis - translation by vector $\begin{pmatrix} 0 \\ 7 \end{pmatrix}$.	
	Range: $f(x) \geq 7$

Solving Equations Using Inverse Hyperbolic Functions

Inverse hyperbolic functions can be used to solve equations involving hyperbolic functions. This can be done by accessing them on your calculator, as would be done with trigonometric functions, or by using their logarithmic forms to find exact values to solutions.

Example 4: Solve $9\sinh(x) = 5\cosh(x)$.

Rearrange by dividing both sides by $\cosh(x)$ and then by 9. Notice $\cosh(x) \geq 1$ so $\cosh(x) \neq 0$ for all x.	$\frac{\sinh(x)}{\cosh(x)} = \frac{5}{9}$
Use the identity $\frac{\sinh(x)}{\cosh(x)} = \tanh(x)$ and then rearrange to get an expression in terms of $\text{artanh}(x)$.	$\tanh(x) = \frac{5}{9}$ $x = \text{artanh}\left(\frac{5}{9}\right)$
Use the logarithmic form of $\text{artanh}(x)$ to find the solution in exact form.	$x = \frac{1}{2}\ln\left(\frac{1 + {}^5/_9}{1 - {}^5/_9}\right)$ $x = \frac{1}{2}\ln\left(\frac{{}^{14}/_9}{{}^4/_9}\right)$ $x = \frac{1}{2}\ln\left(\frac{7}{2}\right)$