Hyperbolic Functions II Cheat Sheet

Hyperbolic Identities

Just as there are identities linking the trigonometric functions together, there are similar identities linking hyperbolic functions together. The hyperbolic identities can all be derived from the trigonometric identities using **Osborn's rule**. This rule states that each circular trigonometric function should be replaced with the corresponding hyperbolic function, and for each $\sinh^2(x)$ term, an extra negative is required. For instance,

$$\cos^2(x) + \sin^2(x) \equiv 1.$$

Replacing all the circular trigonometric functions with their corresponding hyperbolic functions,

$$\cosh^2(x) + \sinh^2(x) \equiv 1.$$

As there is a $\sinh^2(x)$ term, a negative sign is needed before it,

$$\cosh^2(x) + \sinh^2(x) \equiv 1.$$

This is given in the formula booklet.

For the AS Further Maths course, knowledge of the above identity and its proof is required, which is shown in Example 1. Another identity to be familiar with is the definition of $\tanh(x)$:

$$\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)}.$$

Example 1: **a)** Prove that $\cosh^2(x) - \sinh^2(x) \equiv 1$. **b)** Hence solve the equation $2\cosh^2(x) + \sinh(x) = 5$, giving your answers in logarithmic form.

a) As with proofs for trigonometric identities, start with the more complicated side, which is the LHS in this case. Write the hyperbolic functions in their exponential forms.	$\text{LHS} = \cosh^2(x) - \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$
Expand the brackets and simplify to arrive at the RHS.	$= \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) - \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)$ $= \frac{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})}{4}$ $= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$ $= \frac{4}{4} = 1 = \text{RHS}$
b) Use the identity from part **a** to write the equation in terms of only $\sinh(x)$ terms.	$2(1 + \sinh^2(x)) + \sinh(x) = 5$
Expand and simplify to achieve a quadratic equation that can be solved.	$2 + 2\sinh^2(x) + \sinh(x) = 5$ $2\sinh^2(x) + \sinh(x) - 3 = 0$
Solve the quadratic equation.	$(\sinh(x) - 1)(2\sinh(x) + 3) = 0$ $\sinh(x) = 1, -\frac{3}{2}$ $x = \text{arsinh}(1), \text{arsinh}\left(-\frac{3}{2}\right)$
Use the fact that $\text{arsinh}(x) = \ln(x + \sqrt{x^2+1})$ to leave the final answers in logarithmic form, as required. It can be useful to check answers using a calculator by substituting the acquired values of x back into the equation.	$x = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln(1 + \sqrt{2})$ $x = \ln\left(-\frac{3}{2} + \sqrt{\left(-\frac{3}{2}\right)^2 + 1}\right) = \ln\left(-\frac{3}{2} + \sqrt{\frac{13}{4}}\right) = \ln\left(\frac{\sqrt{13} - 3}{2}\right)$

Double Angle Identities (A-Level Only)

Additionally, there are hyperbolic identities that are like the double angle formulae for $\sin(x)$ and $\cos(x)$. These can also be derived by Osborne's rule.

$$\sinh(2x) \equiv 2\sinh(x)\cosh(x)$$

$$\cosh(2x) \equiv \cosh^2(x) + \sinh^2(x)$$

$$\equiv 2\cosh^2(x) - 1$$

$$\equiv 2\sinh^2(x) + 1$$

The identities will be provided in the formula book, but questions may ask to prove them and use them to solve equations.

Example 2: Using the definitions of $\sinh(x)$ and $\cosh(x)$, prove that $\cosh^2(x) + \sinh^2(x) \equiv \cosh(2x)$.

Rewrite one side of the equation in terms of exponentials. The LHS has been chosen here.	$\text{LHS} = \cosh^2(x) + \sinh^2(x) = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$
Expand the brackets and simplify to arrive at the RHS.	$= \left(\frac{e^{2x} + 2 + e^{-2x}}{4}\right) + \left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)$ $= \frac{e^{2x} + 2 + e^{-2x} + (e^{2x} - 2 + e^{-2x})}{4}$ $= \frac{e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}}{4}$ $= \frac{2e^{2x} + 2e^{-2x}}{4}$ $= \frac{e^{(2x)} + e^{-(2x)}}{2}$ $= \cosh(2x) = \text{RHS}$

Identities Involving Reciprocals of Hyperbolic Functions (A Level Only)

By dividing through $\cosh^2(x) - \sinh^2(x) \equiv 1$ by either $\cosh^2(x)$ or $\sinh^2(x)$ respectively, it is possible to acquire two further identities:

$$\text{sech}^2(x) \equiv 1 - \tanh^2(x),$$
$$\text{cosech}^2(x) \equiv \coth^2(x) - 1.$$

Note that this is identical to how $\sec^2 x \equiv 1 + \tan^2 x$ and $\text{cosec}^2 x \equiv \cot^2 x + 1$ are derived from $\cos^2 x + \sin^2 x = 1$. Alternatively, Osborn's rule can be used again.

Example 3: **a)** Prove that $\coth(x) + \text{cosech}(x) \equiv \coth\left(\frac{x}{2}\right)$. **b)** Use the result from part **a** to find the solutions to $\coth\left(\frac{3x}{2}\right) + \text{cosech}\left(\frac{3x}{2}\right) = p$ in the form $y\ln(k)$, where y is a rational constant and k is an expression in terms of p. **c)** State the values of p for which the equation is defined.

a) Rewrite the LHS in terms of the standard hyperbolic functions (an alternative method would be to write the hyperbolic functions in their exponential forms).	$\text{LHS} = \coth(x) + \text{cosech}(x) = \frac{\cosh(x) + 1}{\sinh(x)}$
Keeping in mind the desired result of $\coth\left(\frac{x}{2}\right)$, rewrite the expression in terms of $\frac{x}{2}$, using the double angle identities.	$= \frac{\left(2\cosh^2\left(\frac{x}{2}\right) - 1\right) + 1}{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}$
Simplify to arrive at the RHS.	$= \frac{2\cosh^2\left(\frac{x}{2}\right)}{2\sinh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{2}\right)}$ $= \frac{\cosh\left(\frac{x}{2}\right)}{\sinh\left(\frac{x}{2}\right)}$ $= \coth\left(\frac{x}{2}\right) = \text{RHS}$
b) Rewrite the equation given in the question in terms of $coth$.	$\coth\left(\frac{3x}{2}\right) + \text{cosech}\left(\frac{3x}{2}\right) = \coth\left(\frac{3x}{4}\right) = p$
Using the fact that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, rewrite the LHS in exponential form by taking its reciprocal and substituting $\frac{3x}{4}$ for x.	$\frac{e^{\frac{3x}{4}} + e^{-\frac{3x}{4}}}{e^{\frac{3x}{4}} - e^{-\frac{3x}{4}}} = p$
Rearrange the equation and simplify to solve for x. Note that line four here involves multiplying through by $e^{\frac{3x}{4}}$ to leave only one term involving x.	$e^{\frac{3x}{4}} + e^{-\frac{3x}{4}} = p\left(e^{\frac{3x}{4}} - e^{-\frac{3x}{4}}\right)$ $e^{\frac{3x}{4}} + e^{-\frac{3x}{4}} = pe^{\frac{3x}{4}} - pe^{-\frac{3x}{4}}$ $(1+p)e^{-\frac{3x}{4}} = (p-1)e^{\frac{3x}{4}}$ $(1+p) = (p-1)e^{\frac{3x}{2}}$ $e^{\frac{3x}{2}} = \frac{1+p}{p-1}$ $\frac{3x}{2} = \ln\left(\frac{1+p}{p-1}\right)$ $x = \frac{2}{3}\ln\left(\frac{1+p}{p-1}\right)$
c) Identify that the restriction on x is that $\ln(m)$ is only valid for $m > 0$.	$x = \frac{2}{3}\ln\left(\frac{1+p}{p-1}\right)$ is defined for $\frac{1+p}{p-1} > 0$
Solve the inequality by multiplying through by $(p-1)^2$ and finding the values of p for which x is greater than 0.	$(1+p)(p-1) > 0$ $p < -1, p > 1$