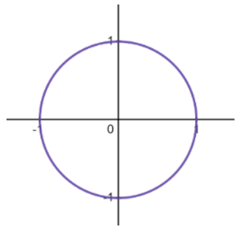
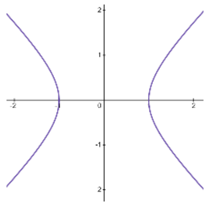
Hyperbolic Functions I Cheat Sheet

Introduction to Hyperbolic Functions



As seen before in the maths syllabus, trigonometric functions are sometimes called circular functions. This is because a point on a unit circle at an angle θ to the positive x-axis has Cartesian coordinates $(\cos\theta, \sin\theta)$.



Another type of curve, a hyperbola, has an equation of the form $x^2 - y^2 = 1$. The Cartesian coordinates for a point on this curve are given by $(\cosh(x), \sinh(x))$. Trigonometric and hyperbolic functions are in some ways related, and analogies between the two can be used to understand definitions and identities.

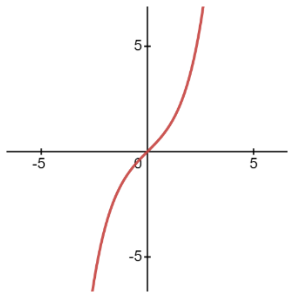
Defining Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

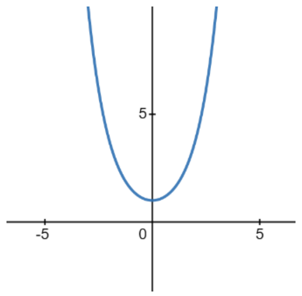
$\tanh(x)$ can be defined in analogy with the trigonometric definition of $\tan(x)$:

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

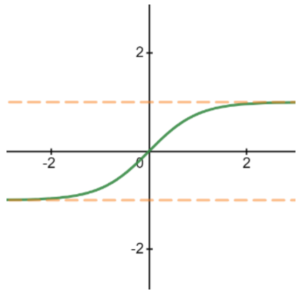
Graphs of Hyperbolic Functions



$\sinh(x)$



$\cosh(x)$



$\tanh(x)$

Domains of Hyperbolic Functions (A Level Only)

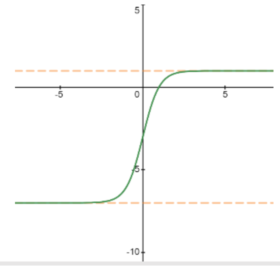
Function	Domain	Range
$\sinh(x)$	$x \in \mathbb{R}$	$f(x) \in \mathbb{R}$
$\cosh(x)$	$x \in \mathbb{R}$	$f(x) \geq 1$
$\tanh(x)$	$x \in \mathbb{R}$	$-1 < f(x) < 1$

The domain and ranges of each of the hyperbolic functions are shown in the table below.

It is also required to be able to work with transformations of the graphs, and to be able to calculate the domain and ranges of these transformations.

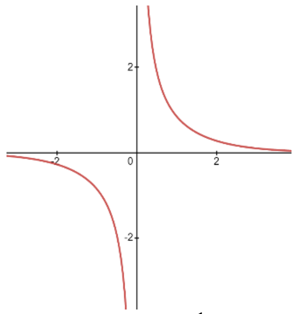
Example 1: Given that $f(x) = 4\tanh(x) - 3$ for $x \in \mathbb{R}$,

a) Sketch the graph of $y = f(x)$
b) State the range of $f(x)$.

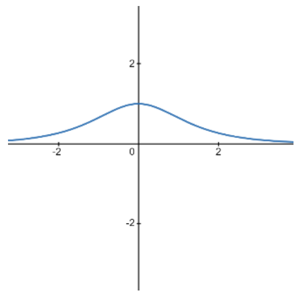
a.) The function $\tanh(x)$ needs to firstly be enlarged by a scale factor of 4 parallel to the y-axis. Then it needs to be translated by the vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$.	
b.) The asymptotes have shifted due the translation, giving us our new range.	$-7 < f(x) < 1$

Reciprocal Hyperbolic Functions (A Level Only)

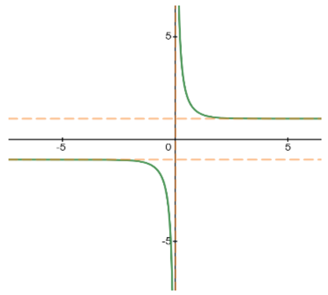
The reciprocal hyperbolic functions are defined in the same way as reciprocal trigonometric functions:



$\mathrm{cosech}(x) = \frac{1}{\sinh(x)}$



$\mathrm{sech}(x) = \frac{1}{\cosh(x)}$



$\coth(x) = \frac{1}{\tanh(x)}$

Domain and Range of Reciprocal Hyperbolic Functions (A Level Only)

Function	Domain	Range
$\mathrm{cosech}(x)$	$x \neq 0$	$f(x) \neq 0$
$\mathrm{sech}(x)$	$x \in \mathbb{R}$	$0 < f(x)$
$\coth(x)$	$x \neq 0$	$f(x) < -1$ or $f(x) > 1$

Differentiating Hyperbolic Functions (A Level Only)

$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\mathrm{cosech}(x) = -\mathrm{cosech}(x)\coth(x)$$

$$\frac{d}{dx}\cosh(x) = \sinh(x) \qquad \frac{d}{dx}\mathrm{sech}(x) = -\mathrm{sech}(x)\tanh(x)$$

$$\frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x) \qquad \frac{d}{dx}\coth(x) = -\mathrm{cosech}^2(x)$$

These results are derived using the exponential definitions of the hyperbolic functions.

Example 2: Show that $\frac{d}{dx}\sinh(x) = \cosh(x)$

Use the exponential definition of $\sinh(x)$	$y = \sinh(x) = \frac{e^x - e^{-x}}{2}$
Differentiate and then equate to the definition of $y = \cosh(x)$	$\frac{dy}{dx} = \frac{e^x - (-e^{-x})}{2}$ $= \frac{e^x + e^{-x}}{2}$ $= \cosh(x)$

Integrating Hyperbolic Functions (A Level Only)

$$\int \sinh(x)\,dx = \cosh(x) + c$$

$$\int \cosh(x)\,dx = \sinh(x) + c$$

$$\int \tanh(x)\,dx = \ln\{\cosh(x)\} + c$$

As with the derivatives, these results can be derived from the exponential definitions of the hyperbolic functions.

Often, the approaches used for integrating trigonometric functions can be used for integrating hyperbolics. This includes the use of identities.

Example 3: Show that $\int \tanh(x)\,dx = \ln\{\cosh(x)\} + c$.

Use the definition of $\tanh(x)$ in terms of $\sinh(x)$ and $\cosh(x)$.	$\int \tanh(x)\,dx = \int \frac{\sinh(x)}{\cosh(x)}\,dx$
This integral is in the following form: $\int \frac{f'(x)}{f(x)}\,dx$ This is a standard integral with the following solution. $\ln\{f(x)\}$ Therefore, it can be integrated directly.	$\int \tanh(x)\,dx = \ln\{\cosh(x)\} + c$

Additionally, sometimes it may be more useful to convert to the exponential definitions of the hyperbolic functions for some integration problems.

Example 4: Find $\int e^x \cosh(x)\,dx$

Use the definition of $\cosh(x)$ in exponential form to rewrite the integral.	$\int e^x\cosh(x)\,dx = \int e^x\left(\frac{e^x + e^{-x}}{2}\right)dx$
The simplified integral can then be evaluated.	$= \int \frac{e^{2x} + 1}{2}\,dx$ $= \frac{1}{2}(e^{2x} + x) + c$