Further Vectors II Cheat Sheet

AQA A Level Further Maths: Core

Vector Product (A level Only)

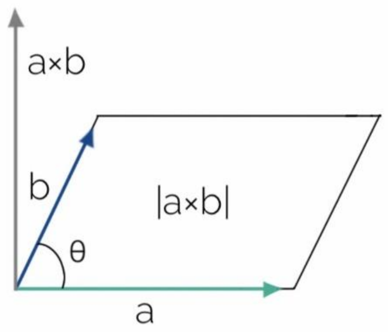
The dot product of two vectors produces a scalar quantity. There is another way to 'multiply' vectors which gives a third vector. It is known as the vector or cross product. It is written as $\boldsymbol{a} \times \boldsymbol{b}$.

The vector product has the following properties:

- $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta$
- $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$
- In component form: $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_y b_z - b_y a_z \\ a_z b_x - b_z a_x \\ a_x b_y - b_x a_y \end{pmatrix}$

Since $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\boldsymbol{\theta}$, the area of a triangle with two sides $\boldsymbol{a}$ and $\boldsymbol{b}$ can be calculated using the cross product:

$$A = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$$

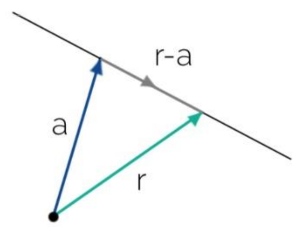


If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{a} \times \boldsymbol{b} = 0$, this follows from noting that if they are parallel then $\theta = 0$ so $\sin\theta = 0$.

This makes the vector product useful for writing the equation of a straight line:

$$(\boldsymbol{r} - \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{0},$$

Here, $\boldsymbol{a}$ is a point on the line and $\boldsymbol{b}$ is a vector parallel to the line. $\boldsymbol{r} - \boldsymbol{a}$ is parallel to $\boldsymbol{b}$ for points which are on the line, hence the cross product is zero.



Example 1: A triangle is formed by the origin, (1,2,6) and (3,4,5). Find the area of the triangle.

Begin by calculating the vectors of two sides of this triangle. Since one of the sides is the origin, the vectors for the two other points are the position vectors.	$\boldsymbol{a} = \begin{pmatrix}1\\2\\6\end{pmatrix}, \boldsymbol{b} = \begin{pmatrix}3\\4\\5\end{pmatrix}$
The cross product can now be calculated, and the modulus can be taken.	$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix}10-24\\18-5\\4-6\end{pmatrix} = \begin{pmatrix}-14\\13\\-2\end{pmatrix}$ $\Rightarrow |\boldsymbol{a} \times \boldsymbol{b}| = \sqrt{(-14)^2 + 13^2 + (-2)^2} = \sqrt{369}$
Now, the formula above can be used to find the area.	$\frac{1}{2}\sqrt{369}$

Geometry of Lines

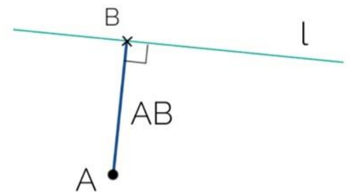
Given two lines in 3D: $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$, $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu \boldsymbol{b}_2$, they intersect if there is a point for which $\boldsymbol{r}_1 = \boldsymbol{r}_2$. Otherwise, they are either parallel or skew. If $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are parallel, then the lines must also be.

Example 2: The line ℓ_1 is given by the equation $\boldsymbol{r}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ and the line ℓ_2 is given by $\boldsymbol{r}_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\3\\1\end{pmatrix}$, where λ and μ are free parameters, do they intersect?

The problem is set up by setting $\boldsymbol{r}_1 = \boldsymbol{r}_2$. This gives 3 simultaneous equations with two unknowns.	$0 + \lambda = 1 + 2\mu\ (1)$ $1 = 1 + 3\mu\ (2)$ $\lambda = 2 + \mu\ \ (3)$
From (2) that, $\mu = 0$ can be used to find the value of λ in (1) and (3). Since we find a contradiction, there is no solution, meaning that the lines do not intersect.	$(2) \Rightarrow \mu = 0$ $\Rightarrow \lambda = 2$ using (3), but $\lambda = 1$ using (1). $\Rightarrow$ No intersection.

Shortest Distance from a Point to a Line

Given a line ℓ and a point A, the shortest distance between A and ℓ can be found. First, B, the point on ℓ which is closest to A, is found by noting that the vector $\boldsymbol{AB}$ must be perpendicular to ℓ. Having found B, the distance can be found as the modulus of $\boldsymbol{AB}$.



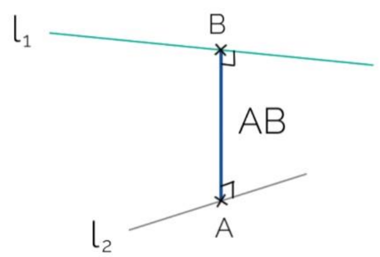
Example 3: The line ℓ is given by $\boldsymbol{r}_1 = \begin{pmatrix}1\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\4\\0\end{pmatrix}$, where λ is a free parameter, what is the shortest distance from ℓ to the point P, $(-1,1,0)$?

The vector from the point to the line can be written down immediately.	$\begin{pmatrix}-1-(1+\lambda)\\1-(1+4\lambda)\\0-3\end{pmatrix} = \begin{pmatrix}-2-\lambda\\-4\lambda\\-3\end{pmatrix}$
When this vector is shortest it is perpendicular to $\begin{pmatrix}1\\4\\0\end{pmatrix}$ so the dot product is zero.	$\begin{pmatrix}1\\4\\0\end{pmatrix} \cdot \begin{pmatrix}-2-\lambda\\-4\lambda\\-3\end{pmatrix} = 0$ $-2 - \lambda - 16\lambda - 3 = -5 - 17\lambda = 0$
λ can be calculated for the point on ℓ closest to P, andh can then be used to find the shortest vector from P to ℓ.	$\lambda = -\frac{5}{17}$ $\begin{pmatrix}-2+\frac{5}{17}\\-4\times-\frac{5}{17}\\-3\end{pmatrix} = \begin{pmatrix}-\frac{29}{17}\\\frac{20}{17}\\-3\end{pmatrix}$
The modulus of this vector can be taken to find the distance.	$\sqrt{\left(\frac{-29}{17}\right)^2 + \left(\frac{20}{17}\right)^2 + (-3)^2} = \sqrt{\frac{226}{17}}$

Shortest Distance Between Two Lines

The shortest distance between two lines is found using a similar idea. The shortest possible vector from one line to the other must be perpendicular to both lines. The points A and B are found by using this fact to set up simultaneous equations.

If the two lines are parallel, then the distance between them can be found more easily by choosing A arbitrarily then finding the shortest distance from A to the line.



Example 4: The line ℓ_1 is given by the equation $\boldsymbol{r}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$ and the line ℓ_2 is given by $\boldsymbol{r}_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\3\\1\end{pmatrix}$, where λ and μ are free parameters, what is the shortest distance from ℓ_1 to ℓ_2?

The vector from ℓ_1 to ℓ_2 can be found as $\boldsymbol{r}_1 - \boldsymbol{r}_2$.	$\begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} - \mu\begin{pmatrix}2\\3\\1\end{pmatrix} = \begin{pmatrix}-1+\lambda-2\mu\\-3\mu\\-2+\lambda-\mu\end{pmatrix}$
When this vector is shortest it is perpendicular to $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}2\\3\\1\end{pmatrix}$.	$\begin{pmatrix}-1+\lambda-2\mu\\-3\mu\\-2+\lambda-\mu\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 0 \Rightarrow -3 + 2\lambda - 3\mu = 0$ $\begin{pmatrix}-1+\lambda-2\mu\\-3\mu\\-2+\lambda-\mu\end{pmatrix} \cdot \begin{pmatrix}2\\3\\1\end{pmatrix} = 0 \Rightarrow -4 + 3\lambda - 14\mu = 0$
The simultaneous equations can be solved to give the values of λ and μ which give the shortest vector $\boldsymbol{r}_1 - \boldsymbol{r}_2$.	$-3 + 2\lambda - 3\mu = 0 \Rightarrow \mu = -1 + \frac{2}{3}\lambda$ $\Rightarrow -4 + 3\lambda - 14\left(1 + \frac{2}{3}\lambda\right) = 0 = -18 - \frac{19}{3}\lambda$ $\Rightarrow \lambda = -\frac{54}{19}, \mu = -\frac{55}{19}$
The modulus of $\boldsymbol{r}_1 - \boldsymbol{r}_2$ with these values of λ and μ is the answer.	$\sqrt{\left(-1 - \frac{54}{19} + \frac{110}{19}\right)^2 + \left(\frac{165}{19}\right)^2 + \left(-2 - \frac{54}{19} + \frac{55}{19}\right)^2} = \sqrt{83}$

Geometry of Planes (A Level Only)

Intersection of a Line and a Plane

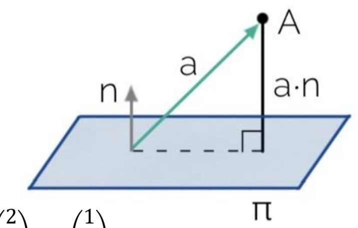
In 3D, a plane and a line will always intersect at a point unless the line is parallel to the plane. The point of intersection can be found most easily using the cartesian equation for the plane.

Example 5: The line ℓ is given by $\boldsymbol{r} = \begin{pmatrix}0\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\1\end{pmatrix}$, the plane π is given by $x + 2y + z = 10$. At what point do they intersect?

Begin by substituting expressions for the coordinates of points on the line into the equation for the plane.	$x = 2\lambda, y = 2 + 4\lambda, z = 1 + \lambda$ $\Rightarrow 2\lambda + 2 + 4\lambda + 1 + \lambda = 10$ $\Rightarrow 7\lambda + 3 = 10$
Solving this gives the value of λ for the point of intersection.	$\lambda = 1$ They intersect at $\begin{pmatrix}2\\6\\2\end{pmatrix}$

Shortest Distance from a Point to a Plane

The shortest distance from a point A, to a plane, π, is most easily found by projecting (taking the scalar product) any vector from A to π (labelled $\boldsymbol{a}$) onto the unit normal vector, $\hat{\boldsymbol{n}}$. This gives the amount of $\boldsymbol{a}$ in the direction of $\boldsymbol{n}$. Since this is perpendicular to the plane, it must be the shortest distance.



Example 6: a.) The plane π is given by the equation $\boldsymbol{r} = \begin{pmatrix}1\\1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$, where λ and μ are free parameters. Find the shortest distance from the point P $(-1,2,-5)$ to π. b.) Given that $\begin{pmatrix}5\\4\\5\end{pmatrix}$ is a point lying on π, find the point of intersection of π with the line ℓ given by $3x + 1 = 2 - y = \frac{z}{3}$.

a.) First, the normal vector for the plane is calculated. This can be done quickly using the vector product.	$\boldsymbol{n} = \begin{pmatrix}2\\3\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3-0\\1-2\\0-3\end{pmatrix} = \begin{pmatrix}3\\-1\\-3\end{pmatrix}$
Next, the unit normal vector $\hat{\boldsymbol{n}}$, is found by dividing $\boldsymbol{n}$ by $|\boldsymbol{n}|$.	$|\boldsymbol{n}| = \sqrt{9+1+9}$ $\Rightarrow \hat{\boldsymbol{n}} = \frac{1}{\sqrt{19}}\begin{pmatrix}3\\-1\\-3\end{pmatrix}$
A vector $\boldsymbol{a}$ from π to the point is needed. The point at $\begin{pmatrix}1\\1\\2\end{pmatrix}$ is clearly on π so we can use this to find $\boldsymbol{a}$.	$\boldsymbol{a} = \begin{pmatrix}-1\\2\\-5\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}-2\\1\\-7\end{pmatrix}$
Now projecting $\boldsymbol{a}$ onto $\hat{\boldsymbol{n}}$ gives the shortest distance from P to π.	$\boldsymbol{a} \cdot \hat{\boldsymbol{n}} = \frac{1}{\sqrt{19}}\begin{pmatrix}-2\\1\\-7\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\-3\end{pmatrix}$ $= \frac{1}{\sqrt{19}}(-6 - 1 + 21) = \frac{14}{\sqrt{19}}$
b.) This part is most quickly solved with the cartesian equation for π and the vector equation for ℓ. Using $\boldsymbol{n}$ and the point $\begin{pmatrix}5\\4\\5\end{pmatrix}$, the cartesian equation for π is found.	From n, $\pi = 3x - y - 3z + d = 0$, where $d = -r \cdot n$ $d = -\begin{pmatrix}5\\4\\5\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\-3\end{pmatrix}$ $3x - y - 3z + 4 = 0$
The equation for ℓ is converted into vector form. Then the same method as in example 5 can be used.	$\sigma = 3x + 1 = 2 - y = \frac{z}{3}$ $\Rightarrow x = -\frac{1}{3} + \frac{\sigma}{3}, y = 2 - \sigma, z = 3\sigma$ $\Rightarrow \boldsymbol{r} = \begin{pmatrix}-\frac{1}{3}\\2\\0\end{pmatrix} + \sigma\begin{pmatrix}\frac{1}{3}\\-1\\3\end{pmatrix}$
The expressions for x, y and z are substituted into the cartesian equation for π.	$3\left(-\frac{1}{3} + \frac{\sigma}{3}\right) - (2 - \sigma) - 3\sigma = -4$ $\sigma = 1$
This value is used in the vector equation for ℓ to give the point of intersection.	$\boldsymbol{r}_{intersection} = \begin{pmatrix}-\frac{1}{3}\\2\\0\end{pmatrix} - 1\begin{pmatrix}\frac{1}{3}\\-1\\3\end{pmatrix} = \begin{pmatrix}-\frac{2}{3}\\3\\-3\end{pmatrix}$