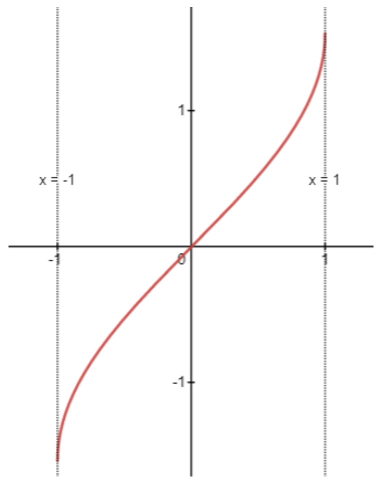
Further Calculus IV Cheat Sheet

Differentiating Inverse Trigonometric Functions (A Level Only)

To find the derivatives of inverse trigonometric functions, implicit differentiation is used. In order for a function to have an inverse, it must have a 1-to-1 mapping between the domain and the range. For example consider $\sin(x)$, its range is $[-1,1]$ and so its inverse, $\sin^{-1}(x)$, is defined for $x \in [-1,1]$.

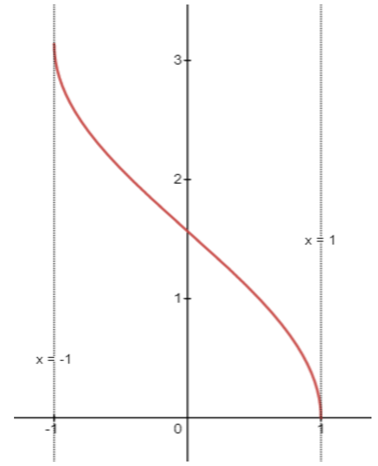
The graph of $y = \sin^{-1}(x)$ is shown to the right. Although this function is defined at $x = \pm 1$, the gradient at these points is infinite, and therefore undefined. Thus, the resitrction $|x| < 1$ is taken when finding the derivative of this function.



The graph of $y = \sin^{-1}(x)$. This function has domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Example 1: Given $|x| < 1$, find $\frac{dy}{dx}$ for $y = \sin^{-1}(x)$.

Apply the sine function to find x in terms of y. This is now differentiable via implicit differentiation.	$y = \sin^{-1}(x) \Rightarrow \sin(y) = x.$
Differentiate this equation with respect to x. The aim is to have an expression for $\frac{dy}{dx}$ in terms of x.	Using the chain rule, $\cos(y)\frac{dy}{dx} = 1$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}.$
Use trigonometric identities to express $\cos(y)$ in terms of x. The range of $\sin^{-1}(x)$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, and since $\cos(y)$ is non-negative in this range, it is justified to take just the positive square root here. Looking at the graph of $\sin^{-1}(x)$ also confirms that the gradient is only ever positive.	$\cos^2(y) = 1 - \sin^2(y)$ $\Rightarrow \cos(y) = \sqrt{1 - \sin^2(y)}.$ It is given that $\sin(y) = x$, and so this expression becomes $\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}} = \frac{1}{\sqrt{1 - x^2}}.$



The graph of $y = \cos^{-1}(x)$. This function has domain $[-1,1]$ and range $[0, \pi]$.

The derivatives of $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are found in a similar way, giving:

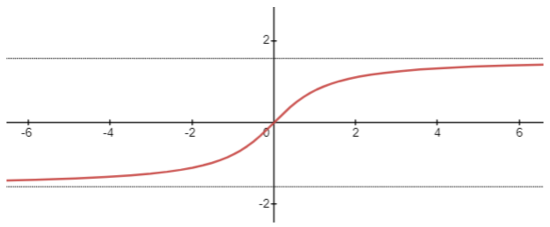
$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1 - x^2}}, |x| < 1 \qquad \frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1 + x^2}.$$

These are given in the formula book. The derivative of $\tan^{-1}(x)$ is defined for all x, since the range of $\tan(x)$ is infinite. These results can also be used within examples which involve the product, chain and quotient rules.

Example 2: Find the derivative of $y = e^{6x}\arctan(2x)$.

There are two functions multiplied together here, so the product rule must be used. Note that $\arctan(x)$ is different notation for $\tan^{-1}(x)$. The chain rule is used to find the derivative of $\arctan(2x)$. $2x$ is used in place of x in the result for its derivative.	Product rule: $\frac{d}{dx}(uv) = u'v + v'u.$ $u = e^{6x},\ u' = 6e^{6x}.$ Using the chain rule $\left(f(g(x))\right)' = f'(g(x))g'(x)$ with $f(x) = \arctan(x), g(x) = 2x$: $v = \arctan(2x),\ v' = \frac{1 \cdot (2x)'}{1 + (2x)^2} = \frac{2}{1 + 4x^2}$ $\therefore u'v + v'u = 6e^{6x}\arctan(2x) + \frac{2e^{6x}}{1 + 4x^2}$ $\therefore \frac{d}{dx}(e^{6x}\arctan(2x)) = e^{6x}\left(6\arctan(2x) + \frac{2}{1 + 4x^2}\right)$

To the right is the graph of $y = \tan^{-1}(x)$. The domain of this function is $(-\infty, \infty)$ since $\tan(x)$ takes on an infinite number of values, unlike $\sin(x)$ and $\cos(x)$. The range of this function is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, with the graph approaching each limit asymptotically. This is due to $\tan(x)$ being undefined at $\pm\frac{\pi}{2}$.



Using Inverse Trigonometric Functions in Integration (A Level Only)

The derivatives of $\sin^{-1}(x)$ and $\tan^{-1}(x)$ are used to evaluate integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}}dx$ and $\int \frac{1}{a^2 + x^2}dx$ respectively. The derivative for $\cos^{-1}(x)$ does not offer any additional help here as it is the negative of the derivative of $\sin^{-1}(x)$. The general results that are used are as follows:

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c, |x| > a \qquad \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

where c is a constant of integration. These are given in the formula booklet. Alongside using these identities, question types may include proving the above results.

Example 3: Use the substitution $x = a\tan(u)$ to prove the result $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$.

Differentiate the substitution with respect to u to find an expression for dx. This is then used to transform the integral. The derivative of $\tan(u)$ is given in the formula booklet.	$x = a\tan(u) \Rightarrow \frac{dx}{du} = a\sec^2(u)$ $\therefore dx = a\sec^2(u)du.$
Rearrange the integrand to express it in terms of u, using any necessary trigonometric identities.	$\frac{1}{a^2 + x^2} = \frac{1}{a^2 + a^2\tan^2(u)} = \frac{1}{a^2} \cdot \frac{1}{1 + \tan^2(u)}.$ Use the trigonometric identity $1 + \tan^2(u) = \sec^2(u).$ $\therefore \frac{1}{a^2 + x^2} = \frac{1}{a^2} \cdot \frac{1}{\sec^2(u)}.$
Combine both steps to complete the substitution. Then, evaluate the integral, not forgetting to add the constant of integration.	$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a^2}\int \frac{1}{\sec^2(u)} \cdot a\sec^2(u)du = \frac{1}{a}\int du$ $= \frac{1}{a}u + c.$
Finally, rewrite u in terms of x to reach the desired result.	$x = \arctan(u) \Rightarrow u = \tan^{-1}\left(\frac{x}{a}\right)$ $\therefore \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$

Note: the result for $\int \frac{1}{\sqrt{a^2 - x^2}}dx$ is similarly proven.

To complete questions using these results, the integrand may need to be rearranged into the desired form

Example 4: Find -

$$\int \frac{1}{\sqrt{-x^2 - 2x + 35}}dx$$

Noting the square root, rearranging the denominator into the form $a^2 - x^2$ will allow for the use of $\sin^{-1}(x)$. Completing the square of this expression will lead to the desired format.	$-x^2 - 2x + 35 = -(x^2 + 2x - 35).$ $x^2 + 2x - 35 = (x + 1)^2 - 1 - 35 = (x + 1)^2 - 36$ $\therefore -x^2 - 2x + 35 = 6^2 - (x + 1)^2$ $\therefore \int \frac{1}{\sqrt{-x^2 - 2x + 35}}dx = \int \frac{1}{\sqrt{6^2 - (x + 1)^2}}dx$
The result can now be applied, with $a = 6$, and replacing x^2 with $(x + 1)^2$, not forgetting the constant of integration.	$\int \frac{1}{\sqrt{6^2 - (x + 1)^2}}dx = \sin^{-1}\left(\frac{x + 1}{6}\right) + c$